Summary

The work of this thesis is focused on the robustness of control laws for spacecraft formation. Robustness in this case refers to the ability of the system to withstand persistent perturbations, and to keep some of the stability characteristics of the unperturbed system.

Analogous to the definition of *practical asymptotic stability* in Chaillet and Loría (2006b), *practical exponential stability* is defined. This definition is more restrictive than its asymptotic counterpart, but has the advantage of an exponentially decaying upper bound of the solution on the considered part of the state space. Lyapunov sufficient conditions are stated, both for general systems and systems which are interconnected on a cascaded structure. Systems can naturally show a cascaded structure, as e.g. a leader follower spacecraft formation, or they can be rewritten into a cascaded structure, which is a common approach for systems with an observer and certainty equivalence controller. Furthermore, a theoretical framework is provided that fits realistic challenges related to spacecraft formation with disturbances. It is shown that the input-to-state property of such systems guarantees some robustness with respect to a class of signals with bounded average-energy, which encompasses the typical disturbances acting on spacecraft formations. Robustness is considered in the sense that solutions are bounded by a converging function of time, up to an offset which is somewhat proportional to the considered average energy of disturbances. The proposed approach allows for a tighter evaluation of the disturbances' influence, which in turns allows for the use of more parsimonious control gains.

With the mathematical background in place, the leader-follower spacecraft formation is modeled. This type of formation is chosen because of its simplicity. It is therefore, in the authors opinion, the type of formation most likely used for real applications in the field of spacecraft formation control in the nearest future. Both a model for relative translation and rotation is derived. The relative translation model is derived in a general setting, where the origin of the frame of reference can be chosen as the center of gravity of the leader spacecraft, or some other convenient point.

Output tracking control laws for the relative translation and rotation models are designed. The follower spacecraft control laws are derived under limited knowledge of the leader spacecraft. It is required that the leader spacecraft can either broadcast its position and attitude, or the follower spacecraft are equipped with devices that can take the necessary measurements. In deriving the control laws, inspiration is taken from the theory for control of robotic manipulators and ocean vehicles, as they are systems with similar properties.

Motivated by the possibly high amplitude/ low energy disturbances acting on the formation, stability of some of the control algorithms with respect to a class of bounded-energy signals are analysed, using the above mentioned framework.

As propulsion systems of spacecraft often do not provide continuous actuation, stability properties of the control algorithms are also analysed when the actuation is quantized or pulse width modulated.

Preface

There are a number of people who should be thanked for helping me in the process of realizing this thesis. First of all I would like to thank my supervisor, Prof. Tommy Gravdahl for giving me the opportunity to be his PhD student, for his encouragement and for seeing the best in my work.

I would also like to thank Prof. Shankar Sastry for welcoming me to UC Berkeley from August 2006 to July 2007. It was a life memorable experience, and by far the best of these four years. During my stay at UC Berkeley I had the pleasure to meet and work with a lot of great people; especially Humberto Gonzalez, Jan Biermeyer, Todd Templeton and Assistant Prof. Jonathan Sprinkle. I learned a lot from the enormous number of man-hours we put into making a Toyota drive autonomously.

The one who undoubtedly has had the most influence on my work, is Assistant Prof. Antoine Chaillet. Not only did he invite me to Supélec, Paris in October/November 2008, a great experience in its own, but he has answered my questions no matter how silly they must seem to him, patiently corrected my mistakes while I steadily have been adding new ones, and improved my technical writing skills. For this, I am truly grateful. He has contributed to the best results of this thesis, and his work has been a major inspiration for my research.

I would like to thank Elena Panteley at Supélec, for her ideas and contributions to our results on stability with respect to a class of signals with bounded average-energy.

Associate Prof. Raymond Kristiansen and Prof. Per Johan Nicklasson at Narvik University College, deserve a big thanks for giving me a kick-start into the research area of spacecraft control, both through mutual publications and for their inspirational work on topics related to this thesis.

I would also like to express my sincere gratitude to friends and colleagues at Department of Engineering Cybernetics at NTNU for creating a good working environment. In particular I have enjoyed the social activities.

Last, but not least, I am grateful to my family and to my girlfriend

Jennifer for their love and support.

Contents

Chapter 1

Introduction

1.1 Motivation

There are several reasons for formations of spacecraft gaining so much interest from the research community in the last decade. The most important is the desire to place measuring equipment further apart than what is possible on a single spacecraft. This is desirable because the resolution of measurements often are proportional to the baseline length, meaning that either a big spacecraft or a formation of smaller, but accurately controlled spacecraft may be used. Big spacecraft that satisfy the demand of resolution are often impractical and both costly to develop and to launch. Smaller spacecraft on the other hand may be standardized and have lower development cost. In addition they may be of a lower collective weight and/or of collective smaller size such that cheaper launch vehicles can be used. There is also the possibility for them to piggy-back with other commercial spacecraft.

What is a formation?

Before we proceed it is important to agree on what is meant by a (spacecraft) formation. We follow Scharf et al. (2003) and Scharf et al. (2004), which define a *formation* as "*a set of more than one spacecraft in which any of the spacecraft dynamic states are coupled through a common control law.*" In particular, "*at least one of the members of the set must:*

1. *track a desired state profile relative to another member, and*

2. *the associated tracking control law must at the minimum depend upon the state of this other member*".

This is sometimes also referred to as an autonomous formation. A *constellation* on the other hand, is "*a set of spacecraft whose states are not dynamically coupled in any way*", Scharf et al. (2003). It should be clear that the global positioning system (GPS) is a constellation as the spacecraft orbit corrections does not require the information about any of the other spacecraft, but are solely based on the individual spacecraft position and velocity.

Applications

Applications can e.g. be automated rendezvous for equipment and fuel delivery. These applications can be considered as special cases, as the need for autonomy is only over a limited time frame. On the other hand, the demands to fault protection and accuracy, are just as high as for other types of missions, due the close proximity of the spacecraft.

Another application is distributed sensors arrays. In deep space, formations will "*enable variable-baseline interferometers and large-scale distributed sensors that can probe the origin and structure of stars and galaxies with high precision*", Scharf et al. (2004). According to the same reference, Earth orbiting formations will "*enable distributed sensing and sparse antenna arrays for applications such as gravitational mapping and interferometric synthetic aperture radar*".

Proposed and ongoing projects demonstrating tandem or formation flight

A list of proposed or ongoing formation flying projects (including tandem flights, which are not autonomous) can be found in e.g. Xu et al. (2007), D'Amico et al. (2005), Gill et al. (2001), Persson et al. (2006) and comprise TerraSAR-X / TanDEM-X, GRACE, the New Millienium Program with EO-1/Landsat, Proba-3, A-Train and Prisma. It is difficult to state which of the projects that will perform true formation flight in the sense of the definition in Section 1.1, as tandem flights often also are described as formations in the literature.

Also, there has been a recent proposal by DARPA which may serve as a motivation for the work of this thesis[1]:

"*The goal of the System F6 program is to demonstrate a radically new space system composed of a heterogeneous network of formation flying or loosely connected small satellite modules that will, working together, provide*

[1] Accessed at "http://www.darpa.mil/tto/programs/system_F6/" 21. August, 2009

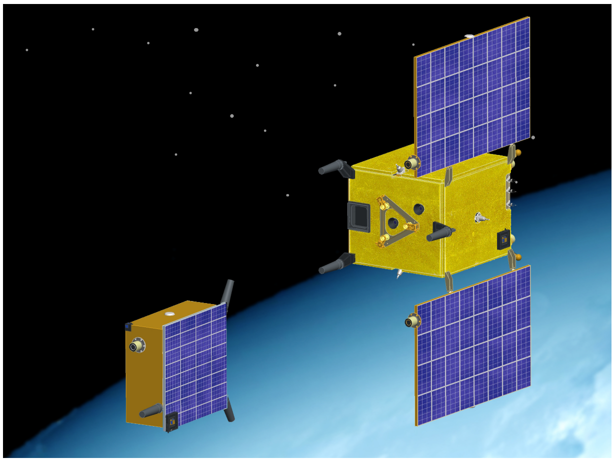

Figure 1.1: Artistic interpretation of the Prisma satellites. Reproduced with courtesy to the Swedish National Space Board (http://www.prismasatellites.se)

at least the same effective mission capability of a large monolithic satellite. Current large space systems used for national security purposes are constrained due to their monolithic architecture. They can be launched only on a small number of large launch vehicles, cannot readily be upgraded and/or reconfigured with new hardware on-orbit, and are risk-intensive, since the unforgiving launch and space environments can result in a total loss of investment with one mistake. The System F6 will partition the tasks performed by monolithic spacecraft (power, receivers, control modules, etc.) and assign each task to a dedicated small or micro satellite. This fractionated space system offers the potential for reduced risk, greater flexibility (e.g. simplified on-orbit servicing, reconfigurability to meet changing mission needs), payload isolation, faster deployment of initial capability, and potential for improved survivability. This program will develop, design, and test new space system architectures and technologies required to successfully decompose a spacecraft into fundamental elements. Such architectures include, but are not limited to, ultra-secure intra-system wireless data communications, wireless power systems, electromagnetic formation flying systems, remote attitude determination systems, structure-less optical and RF arrays, distributed spacecraft computing systems, and reliable, robust, rapidly re-locatable ground systems."

Explicit choices in this thesis

There are a few explicit choices that have been made in this thesis and which should be given the grounds for: We only consider leader-follower type of formation. This is motivated by the above mentioned projects, which for most are formations of two spacecraft, and the fact that autonomous formation control of spacecraft as an engineering problem is in its evolutionary cradle, which prompts for simple solutions.

We use the full nonlinear model of the formation, and do not linearize about a point of reference as is commonly done in the literature on formation flying spacecraft. This choice is taken to be able to handle formations in strongly elliptic orbits and formation with long baseline. Also the required precision of the proposed project, does not allow for severe approximations.

The focus on output feedback in this thesis is motivated by the fact that position and velocity measurements in space may not be easily achieved, e.g. because the formation is outside the coverage of the Global Positioning System (GPS), or because the the spacecraft can not be equipped with the necessary sensors for such measurements due to space constraints or budget limits. Numerical derivatives are not well suited, as they may amplify measurement noise.

Although output feedback will be treated extensively in this thesis, the use of Kalman filters, which can be found very useful for this type of missions (where measurements are correlates with noise) and which are able to provide velocity information from position measurements, have not been considered in the analysis. One of the reasons for this is that Kalman filters have already been thoroughly treated in the literature. More importantly, the main focus of this thesis is on strong stability properties, which may be difficult to achieve for Kalman filters although they may provide the necessary estimates.

1.1.1 Literature review

Practical stability and input-to-state-stable systems

Notice that the term *practical stability* has different meaning in the literature of control theory, see Chaillet and Loría (2006b) for a discussion on this matter. Our understanding of the term is that of Chaillet and Loría (2006b) where the vicinity of the origin to which the solutions converge, may be made arbitrarily small by convenient tuning of some parameters of the system, typically the control gains. This meaning is in fact consistent with the *narrower* stability property referred to in the classic text book on stability (Hahn, 1967, Page 278): *"One does not* [when talking about his definition of practical stability]*, however, insist on the narrower stability property; that is one will not require that the deviation from zero can be made arbitrarily small by a suitable choice of constants".* We stress that ultimate boundedness as defined in Khalil (2002) is a *weaker* property than practical stability.

Input-to-state stability (ISS) is a concept introduced in (Sontag, 1989), which has been thoroughly treated in the literature: see for instance the survey (Sontag, 2007) and references therein. Roughly speaking, this robustness property ensures asymptotic stability, up to a term that is "proportional" to the *amplitude* of the disturbing signal. Similarly, its integral extension, iISS (Sontag, 1998), links the convergence of the state to a measure of the *energy* that is fed by the disturbance into the system. However, in the original works on ISS and iISS, both these notions require that these indicators (amplitude or energy) be finite to guarantee some robustness. In particular, while this concept has proved useful in many control application, ISS may yield very conservative estimates when the disturbing signals come with high amplitude even if their *moving average* is reasonable.

These limitations have already been pointed out and partially addressed in the literature. In Angeli and Nešić (2001), the notions of "Power ISS" and "Power iISS" are introduced to estimate more tightly the influence of the power or moving average of the exogenous input on the *power* of the state. Under the assumption of local stability for the zero-input system, these properties are shown to be actually equivalent to ISS and iISS respectively. Nonetheless, for a generic class of input signals, no hard bound on the state norm can be derived for this work.

Other works have focused on quantitative aspects of ISS, such as (Praly and Wang, 1996), (Grüne, 2002) and (Grüne, 2004). All these three papers solve the problem by introducing a "memory fading" effect in the input term of the ISS formulation. In (Praly and Wang, 1996) the perturbation is first fed into a linear scalar system whose output then enters the right hand side of the ISS estimate. The resulting property is referred to as exp-ISS and is shown to be equivalent to ISS. In (Grüne, 2002) and (Grüne, 2004) the concept of input-to-state dynamical stability (ISDS) is introduced and exploited. In the ISDS state estimate, the value of the perturbation at each time instant is used as the initial value of a one-dimensional system, thus generalizing the original idea of Praly and Wang. The quantitative knowledge of how past values of the input signal influence the system allows, in particular, to guarantee an explicit decay rate of the state for vanishing perturbations.

Relative translational tracking

Research on spacecraft formation control is vast, and makes us unable to serve the enormous amount of literature justice here. We will therefore focus on previous work done on spacecraft formations where a relative position model similar to the one in Chapter 3 are used. For a more thorough treatment of the topic of spacecraft formation control, the interested reader is instead referred to the survey paper Scharf et al. (2004). One of the solutions to the control problem of the relative position model was presented in Queiroz et al. (1999). There, a nonlinear output feedback control law was developed guaranteeing global uniform ultimate boundedness (GUUB) of the position and velocity tracking errors in the presence of unknown spacecraft masses and disturbance force parameters. A filtering scheme was provided, to allow for the use of relative velocity in the controller. A similar result was given in Yan et al. (2000). In Pan and Kapila (2001) the nonlinear tracking control problem for both translation and rotation was presented. The adaptive control law derived, ensure global asymptotic convergence in the presence of unknown mass and inertia of the leader and follower spacecraft. In Wong et al. (2001) a full state feedback adaptive learning control algorithm was developed to give global asymptotic convergence of position and velocity tracking errors, in the presence of periodic disturbances and unknown spacecraft masses. An internal model based approach was taken in Serrani (2003) to design a controller that handles parametric uncertainties and unknown disturbances. The methodology was shown to be robust to persistent disturbances, such as gravitational perturbations. Assuming boundedness of orbital perturbations and the leader control force only, an adaptive controller was designed in Kristiansen et al. (2006b) to prove that the closed-loop system is uniformly semiglobally practically asymptotically stable (USPAS). A velocity filter was used to provide sufficient knowledge about the relative velocity to solve the control problem. These results were extended in Kristiansen et al. (2006a) to also include the case of uncertainty in spacecraft mass.

Relative rotational tracking

The following is a presentation of some of the works done on output control of spacecraft using quaternion measurements. A globally convergent angular velocity observer can be found in Salcudean (1991) and is highly referenced in the later works on output control of spacecraft. In Lizarralde and Wen (1996) a nonlinear filter is used to compensate for missing velocity measurements. The passivity properties of the system are exploited in an output controller so as to achieve asymptotic stabilization of the closed-loop system. A nonlinear quaternion based feedback control law is used in Joshi et al. (1995) to achieve similar stability results. The controller does not depend on system parameters, and therefore robustness to modeling errors and parametric uncertainties are ensured. Two schemes for output attitude tracking are presented in Caccavale and Villani (1999). The schemes are based on results achieved for output control of robot manipulators, see Berghuis and Nijmeijer (1993), but as mentioned in Caccavale and Villani (1999) the extension is not straight forward due to the nonlinear mapping between the orientation variables, the unit quaternions. In Bondhus et al. (2005) output control is applied to the synchronization of a leader/follower formation of spacecraft. Nonlinear observers are used to estimate the angular velocities based on quaternion measurements, and the rotation matrices representing the attitude error between the reference trajectory and the leader and the follower spacecraft are shown to converge to the identity matrix from any initial condition. The tracking control problem of a follower spacecraft with coupled rotational and translational motion is addressed in Wong et al. (2005). Convergence of the position and tracking errors are proven, using only position and attitude orientation measurements. In Tayebi (2006) a spacecraft is stabilized without the use of velocity measurements. A unit quaternion observer is used together with linear feedback in terms of the vector parts of the actual unit quaternion and the estimation error quaternion. Asymptotic stability is proven through Lyapunov analysis. The model of the relative dynamics used in this paper has also been treated in Kristiansen et al. (2006c) and Krogstad et al. (2007). In Kristiansen et al. (2009) a controller was designed which incorporates an approximate-differentiation filter to account for the unmeasured angular velocity. The closed-loop system was shown to be UPAS.

1.2 Contributions and limitations of this thesis

1.2.1 Contributions

In the following the contributions of the work presented in this thesis are summarized. The labels are with reference to the publication list in Section 1.2.2:

In Chapter 2 we present a theoretical contribution consisting of new definitions and theorems of sufficient conditions for nonlinear time-varying systems to be exponentially stable with respect to balls that can be arbitrarily reduced by a convenient tuning. We denote a system satisfying these properties in the whole state-space uniformly globally practically exponentially stable (UGPES). For the sake of completeness, we also discuss uniform semiglobal exponential stability (USES) and uniform semiglobal practical exponential stability (USPES), in which case the domain of attraction is not the whole state-space, but a compact set that can be arbitrarily enlarged. These results were published in vi/.

Furthermore, we provide a theorem of sufficient conditions for a cascaded system to be UGPES, uniformly practically exponentially stable (UPES) or uniformly globally practically asymptotically stable (UGPAS). As many of the disturbances acting on spacecraft are difficult to model, we define a general class of signals with limited excitation *in average*. By explicit knowledge of an ISS Lyapunov function, and in particular its dissipation rate, we are able to identify the class to which it is robust, in the sense that the solutions are bounded by a $\mathcal{KL}$ estimate and a constant (corresponding to the predefined required precision). These results are contained in iv/. The mathematical framework is put forward in Chapter 2.

Most of what is presented in Chapter 3 is based on previous published materials by other authors, e.g. Ploen et al. (2004b). Some new and important properties of the models were however published in iii/. Also, based on the content of Chapter 3, we show in subsequent chapters how different choices of reference frames simplifies the stability analysis of the overall formation, and gives stronger stability results.

Chapter 4 through 6 contain applications of the theory in Chapter 2. In the following, we will therefore summarize how our applications are different from other results in the literature.

In Chapter 4 the stability of a leader/follower formation is analyzed using a controller-observer scheme originally designed for the control of robot manipulators. While, in the nominal case, the solutions of the system are proven to be exponentially convergent to zero, we will show that the steady-state error resulting from external disturbances and lack of measurement can be arbitrarily diminished by a convenient tuning of some controller gains. In fact, based on knowledge on the bounds of the disturbances and the acceptable steady state error, the presented theorems give information on how to pick the controller gains. These results were published in xi/ and vi/.

In Chapter 5 the attitude tracking problem of a leader/follower formation under external disturbances is considered. As opposed to most other papers on the topic, the control of both the leader and follower spacecraft are considered, and the solutions of the system are proved to be exponentially convergent to zero, up to a steady-state error that can be arbitrarily reduced by a convenient tuning of the control gains. The results of this chapter are based on vii/.

In Chapter 6 we show that the overall formation is input-to-state stable (ISS) with respect to an extended disturbance, which from the follower spacecraft point of view not only includes the external disturbances, but also the leader spacecraft reference trajectory. Using the framework of Section 2.5, we find an explicit bound on the tolerable average excitation. The contents of this chapter can also be found in iv/.

Although, propulsion systems of spacecraft often do not provide continuous actuation, stability analysis of such systems have hardly been treated in the literature for systems with nonlinear plants. Chapter 7 is devoted to the analysis of such systems when the actuation is quantized or pulse width modulated. The results of quantized actuation have been published in x/ and ix/.

1.2.2 List of publications

The following list contains the authors publications and recently submitted papers:

Journal papers

i/ Grøtli, E. I., Chaillet, A., Panteley, E., Gravdahl, J. T., 2010a. Robustness of ISS systems to inputs with limited moving average, with application to spacecraft formations. International Journal of Robust and Nonlinear Control. (Submitted).

ii/ Sprinkle, J., Eklund, J. M., Gonzalez, H., Grøtli, E. I., Upcroft, B., Makarenko, A., Uther, W., Moser, M., Fitch, R., Durrant-Whyte, H. and Sastry, S. S., 2009. Model-based design: A report from the trenches of the DARPA Urban Challenge. Software and Systems Modeling 8, 551-556.

iii/ Kristiansen, R., Grøtli, E. I., Nicklasson, P. J. and Gravdahl, J. T., 2007. A model of relative translation and rotation in a leader-follower spacecraft formation. Modeling, Identification and Control 28 (1), 3-13.

Conference papers

iv/ Grøtli, E. I., Chaillet, A., Panteley, E., Gravdahl, J. T., 2010b. Robustness of ISS systems to inputs with limited moving average, with application to spacecraft formations. In: Proc. of the International Conference on Informatics in Control, Automation and Robotics.(Submitted).

v/ Sprinkle, J., Eklund, J. M., Gonzalez, H., Grøtli, E. I., Sanketi, P., Moser, M., and Sastry, S. S., 2010. Recovering Models of a Four-Wheel Vehicle Using Vehicular System Data. In: -. (In preparation).

vi/ Grøtli, E. I., Chaillet, A., Gravdahl, J. T., 2008. Output control of spacecraft in leader follower formation. In: Proc. of the 47th IEEE Conference on Decision and Control. pp. 1030-1035.

vii/ Grøtli, E. I., Gravdahl, J. T., 2008b. Output attitude tracking of formation of spacecraft. In: Proc. of the 17th IFAC World Congress. pp. 2137-2142

viii/ Gonzalez, H., Grøtli, E. I., Templeton, T. R., Biermeyer, J. O., Sprinkle, J. and Sastry, S. S., 2008. Transitioning control and sensing technologies from fully-autonomous driving to driver assistance systems. In: Proc. of Automatisierungs-, Assistenzsysteme und eingebettete Systeme für Transportmittel.

ix/ Grøtli, E. I., Gravdahl, J. T., 2008a. Formation control by quantized output feedback. In: Proc. of the 3rd International Symposium on Formation Flying, Missions and Technologies

x/ Grøtli, E. I., 2007. Analysis of a nonlinear continuous control algorithm, in the case of discontinuous actuation. In: Proc. of the 58th International Astronautical Congress

xi/ Grøtli, E. I., Gravdahl, J. T., 2007. Passivity based controller-observer schemes for relative translation of a formation of spacecraft. In: Proc. of the 26th American Control Conference. pp. 4684-4689

The publications ii/, v/ and viii/ were produced during the same time period, but are outside the scope of this thesis.

1.2.3 Limitations

Only leader-follower type of formations are considered in this thesis. Unless otherwise stated, it is assumed that both the structure and parameters of the models are known.

There is no explicit treatment of the control force saturation. For practical stability, it is assumed that the gains can be chosen sufficiently large to achieve the prespecified precision. This means that for accurate precision, the required actuation forces may become larger than what an actual control system can provide. Furthermore, the spacecraft are considered to be overactuated and that thrust is available in the necessary directions.

All signals are deterministic and without delay, and no explicit concern on how to achieve the sensored information has been taken. Collision avoidance is assumed to be ensured by a supervisory control level, and is considered to be outside the scope of this thesis.

1.2.4 Organization of this thesis

Chapter 2: This chapter makes up the theoretical framework for this thesis. Analogous to the definition of *practical asymptotic stability* in Chaillet and Loría (2006b), we define *practical exponential stability.* This definition is more restrictive than its asymptotic counterpart, but is a stronger result in the sense that the solutions are bounded by an exponentially decaying function on the considered part of the state space. Lyapunov sufficient conditions are stated, both for general systems and systems which are interconnected on cascaded structure. Systems can naturally show a cascaded structure, as e.g. a leader follower spacecraft formation, or they can be rewritten to the desired structure such as systems with both controller and observer. Furthermore, we study the robustness of a class of nonlinear systems with respect to a certain class of signals. Such signals are typically external disturbances, but from a follower spacecraft point of view, these signals may also be the reference trajectory of the leader spacecraft. Reference trajectories often belong to the considered set of signals.

Chapter 3: Here, the leader-follower spacecraft formation is modeled. This type of formation is chosen because of its simplicity. It is therefore, in the author's opinion, the type of formation most likely used for real applications in the relatively new field of spacecraft formation control. Both a model for relative translation and rotation is derived. The relative translation model is derived in a general setting, where we can choose the origin of the frame of reference as center of gravity of the leader spacecraft or some other convenient point.

Chapter 4: This chapter concerns output tracking of relative translation. The follower spacecraft control law is derived under limited knowledge of the leader spacecraft. It is required that the leader spacecraft can either broadcast its position, or the follower spacecraft are equipped with devices that can give the necessary measurements. In addition, it is assumed that the control action and disturbances acting on the leader spacecraft is upper bounded. In deriving the control laws we make use of the theory for control of robotic manipulators and ocean vehicles, as they are systems with similar properties.

Chapter 5: This chapter is concerned with output attitude tracking. As opposed to the translational case in Chapter 4, we derive control laws for both the leader- and the follower spacecraft. The error dynamics is naturally on a cascaded structure, and we apply the theorem for UPES derived in Chapter 2 in the analysis.

Chapter 6: Here we analyse the controllers of a spacecraft formation, using the framework of Section 2.5. Our application show that the framework is not only useful for systems perturbed by certain disturbances, but we also show that the reference trajectory of the leader spacecraft can be seen as a disturbance from the follower spacecraft point of view.

Chapter 7: As propulsion systems of spacecraft often do not provide continuous actuation, this chapter is devoted to the analysis of such systems when the actuation is quantized or pulse width modulated.

1.3 Mathematical preliminaries

1.3.1 Notation

- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ denote the set of all nonnegative integers, real numbers, complex numbers and quaternions, respectively. We use $\mathbb{R}_{\geq 0}$ to denote all nonnegative real numbers, and $\mathbb{N}_{\geq N}$ to denote all integers greater or equal to N. $\lfloor \cdot \rfloor$ denote the floor function, i.e. $\lfloor x \rfloor$ is the largest integer not greater than x.

- The time derivatives of a function $x(t)$ are denoted $\dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t}$, $\ddot{x} := \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$, ..., $x^{(n)} := \frac{\mathrm{d}^{(n)} x}{\mathrm{d}t^{(n)}}$

- The p-norm of a vector $x \in \mathbb{R}^n$ is defined as $|x|_p := (\sum_{i=1}^{n} |x_i|^p)^{1/p}$, for $1 \leq p < \infty$ and $|x|_\infty := \max_i |x_i|$. Of notational simplicity we define the Euclidean vector norm as $|x| := |x|_2 = (x^\top x)^{1/2}$.

- We use $|\cdot|$ for the induced L_2 norm of matrices.
- We use $\mathrm{diag}(\cdot, ..., \cdot)$ to denote diagonal- or block diagonal matrices, with the elements within the parenthesis along the diagonal.
- The $\mathcal{L}_p$ and $\mathcal{L}_\infty$ norms of a measurable function $\phi : \mathbb{R} \to \mathbb{R}^n$ are defined as $||\phi||_p := (\int_{t_0}^{\infty} ||\phi(t)||^p \mathrm{d}t)^{1/p}$, and $||\phi||_\infty := \mathrm{ess\,sup}_{t\geq 0} |\phi(t)|$.
- The *open ball* in $\mathbb{R}^n$ of radius δ about x_0 is defined by $\mathcal{B}_\delta(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \delta\}$. We use $\mathcal{B}_\delta$ for the open ball about the origin, that is $\mathcal{B}_\delta := \{x \in \mathbb{R}^n : |x| < \delta\}$.
- The set $\mathcal{A} \subset \mathbb{R}^n$ is *open* if for any $x \in \mathcal{A}$ there exists a real number δ such that $\mathcal{B}_\delta(x) \subset \mathcal{A}$. $\mathcal{A}$ is *closed* if the complement $(\mathbb{R}^n/\mathcal{A})$ is open. The closure of an open set $\mathcal{A}$ is denoted $\overline{\mathcal{A}}$.
- We use $|\cdot|_{\mathcal{A}}$ to denote the distance-to-set function, that is $|x_1|_{\mathcal{A}} := \inf\{|x_1 - x_2| : x_2 \in \mathcal{A}\}$.
- A set $\mathcal{A}$ is convex if for each $x_1, x_2 \in \mathcal{A}$, $\alpha x_1 + (1-\alpha)x_2 \in \mathcal{A}, \forall \alpha \in [0, 1]$. The closed convex hull of a set $\mathcal{A}$, that is, the smallest closed convex set containing $\mathcal{A}$, is denoted $\overline{co}\mathcal{A}$.
- The function $f : [a, b] \to \mathbb{R}$ is *continuous* if for each $\epsilon > 0$ and each $x \in [a, b]$ there is a $\delta > 0$ such that

$$y \in [a, b] \quad \text{and} \quad |y - x| < \delta \implies |f(y) - f(x)| < \epsilon.$$

- The function $g : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for each $\epsilon > 0$ there is a $\delta > 0$ such that whenever $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ are disjoint intervals in $[a, b]$ we have

$$\sum_{k=1}^{n} \beta_k - \alpha_k < \delta \implies \sum_{k=1}^{n} |g(\beta_k) - g(\alpha_k)| < \epsilon.$$

- A measurable function $u : [t_0, +\infty) \to \mathbb{R}^n$, n positive integer, is said to be *essentially bounded* if $\mathrm{ess\,sup}_{t\in[t_0,+\infty)} |u(t)| < +\infty$, and *locally essentially bounded* if, for any $T > t_0$, $u_{[t_0,T)}$ is essentially bounded, where $u_{[t_0,T)} : [t_0, +\infty) \to \mathbb{R}^n$ is the function given by

$$u_{[t_0,T)}(t) = \begin{cases} u(t) & \text{for all } t \in [t_0, T) \\ 0 & \text{elsewhere.} \end{cases}$$

- A continuous function $\alpha : \mathbb{R}_{\geq} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is strictly increasing and $\alpha(0) = 0$. If, in addition, $\alpha(s) \rightarrow \infty$ as $s \rightarrow \infty$, then α is of class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$). A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, $\beta(\cdot, t) \in \mathcal{K}$ for any $t \in \mathbb{R}_{\geq 0}$, and $\beta(s, \cdot)$ is decreasing and tends to zero as s tends to infinity.
- The maximum and minimum eigenvalue of a matrix A is denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively.
- $I_{n\times n}$ and $0_{n\times n}$ denote the $n\times n$ identity- and zero matrix, respectively.
- Given a vector $\omega = \mathrm{col}(\omega_1, \omega_2, \omega_3)$, the matrix S is the skew-symmetric operator defined as

$$S(\omega) := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

i.e. $S(\omega) = -S^\top(\omega)$. We use SS to denote the set of skew-symmetric matrices.

The notation $\vec{x}$ is used for a *coordinate-free* or *geometric* vector, a quantity of both magnitude $|\vec{x}|$ and direction. By *coordinate-free* we mean that this description does not rely on the definition of any coordinate frame, but obeys the parallelogram law of addition in the three dimensional Euclidean point space, $\mathbb{E}^3$, see Ploen et al. (2004a). In a coordinate frame $\mathcal{F}_e$, the vector $\vec{x}$ can be expressed as a linear combination of the orthogonal unit vectors $\vec{e}_i$, $i \in \{1, 2, 3\}$, by

$$\vec{x} = x_1^e \vec{e}_1 + x_2^e \vec{e}_2 + x_3^e \vec{e}_3,$$

where $x_i = \vec{x} \cdot \vec{e}_i$ are the Cartesian coordinates of $\vec{x}$ in $\mathcal{F}_e$. The time derivative of a vector $\vec{x}$ with reference to $\mathcal{F}_e$ is defined by

$$\frac{{}^e\mathrm{d}}{\mathrm{d}t}\vec{x} := \dot{x}_1^e \vec{e}_1 + \dot{x}_2^e \vec{e}_2 + \dot{x}_3^e \vec{e}_3$$

A *coordinate vector* is another convenient form to describe $\vec{x}$, where the coordinates with respect to a particular coordinate frame, in this case $\mathcal{F}_e$, are written as a column vector:

$$x^e = \mathrm{col}\left(x_1^e, x_2^e, x_3^e\right).$$

The time derivative coordinate vector is represented as:

$$\dot{x}^e = \mathrm{col}\left(\dot{x}_1^e, \dot{x}_2^e, \dot{x}_3^e\right).$$

1.3.2 Rotation matrices and unit quaternions

We use the rotation matrix R_b^a, to transform vectors represented in coordinate frame $\mathcal{F}_a$ to $\mathcal{F}_b$, while preserving the length of the vectors. Rotation matrices are special orthogonal matrices in $\mathbb{R}^{3\times 3}$, that is, they belong to the space

$$SO(3) = \left\{ R \in \mathbb{R}^{3\times 3} \quad | \quad R^\top R = I_{3\times 3}, \det(R) = 1 \right\}.$$

We will repeatedly use the fact that $(R_b^a)^\top = (R_b^a)^{-1} = R_a^b$ (where R_a^b is equivalent to the opposite rotation of R_b^a), that the rotation matrix of a composite rotation is given by the product of the rotation matrices (i.e. $R_c^a = R_b^a R_c^b$), and that

$$\dot{R}_b^a = S\left(\omega_{ab}^a\right) R_b^a.$$

The vector ω_{ab}^a is the angular velocity vector. The subscript denotes the angular velocity of reference frame $\mathcal{F}_b$ relative to frame $\mathcal{F}_a$, where as the superscript shows that the vector is decomposed in frame $\mathcal{F}_a$. When clear from the context, we may leave out the superscript of notational simplicity. Two important properties of the indexed angular velocity representation are $\omega_{ab}^a = -\omega_{ba}^a$ and $\omega_{ac}^a = \omega_{ab}^a + \omega_{bc}^a$. The quaternions are a generalization of the complex numbers, and the set of quaternions, denoted by $\mathbb{H}$, is defined as, see Ma et al. (2004):

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j, \quad \text{with } j^2 = -1$$

and where the set of complex numbers is defined as $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ with $i^2 = -1$. Furthermore, an element of $\mathbb{H}$, that is a quaternion, is of the form

$$Q = \eta + \epsilon_1 i + \epsilon_2 j + \epsilon_3 k$$

with $\eta, \epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$ and $k = ij = -ji$. In this paper we will focus on a subgroup of $\mathbb{H}$, the unit quaternions:

$$\mathbb{S}^3 = \left\{ Q \in \mathbb{H} \quad | \quad |Q|^2 = 1 \right\}. \tag{1.1}$$

The unit quaternions (or Euler parameters) can be used to represent rotation matrices, and this representation has the advantage of avoiding singularities (as opposed to rotation matrices represented with Euler angles). We will in the following use the vector q to represent the quaternions, with its elements being the real elements of Q, i.e. $q = (\eta, \epsilon^\top)^\top$ where

$\epsilon = \mathrm{col}(\epsilon_1, \epsilon_2, \epsilon_3)$. The rotation matrix for the unit quaternions is (see Hughes (1986))

$$R(q) = I_{3\times 3} + 2\eta S(\epsilon) + 2S^2(\epsilon).$$

Therefore, q and $-q$ represents the same orientation. We use $\bar{q}$ to denote the complex conjugate of q, i.e. $\bar{q} = (\eta, -\epsilon^\top)^\top$. The quaternion product between two vectors $q_a = (\eta_a, \epsilon_a^\top)^\top$ and $q_b = (\eta_b, \epsilon_b^\top)^\top$ is defined, see Egeland and Gravdahl (2002), as

$$q_a \otimes q_b = \begin{bmatrix} \eta_a \eta_b - \epsilon_a^\top \epsilon_b \\ \eta_a \epsilon_b + \eta_b \epsilon_a + S(\epsilon_a)\epsilon_b \end{bmatrix}.$$

We define the matrix

$$E(q) = \eta I_{3\times 3} + S(\epsilon).$$

The kinematic differential equation can now be derived as

$$\dot{q} = \frac{1}{2}\begin{bmatrix} -\epsilon^\top \\ E(q) \end{bmatrix}\omega,$$

relating the time derivative of the quaternion to the angular velocity. We will use the notation q_{ab} for the quaternion describing the orientation of a frame $\mathcal{F}_b$ relative to a frame $\mathcal{F}_a$. Perfect tracking in terms of the quaternion error $q_{dl} = \bar{q}_{id} \otimes q_{il}$, where $q_{id}(t)$ represents a possibly time varying reference orientation and q_{id} represents the actual orientation, is achieved when $q_{dl} = \mathrm{col}(\pm 1, 0, 0, 0)$.

Chapter 2

Mathematical Preliminaries

2.1 Practical stability

The formal study of spacecraft formation requires solid theoretical roots. In this chapter, a theoretical framework that fits realistic challenges related to this problem is presented, which is also contained in Grøtli et al. (2008) and Grøtli et al. (2010a). The material highly builds on the work in Chaillet (2006). Indeed, in presence of uncertainties or disturbances, it is often the case that a nominally *asymptotically* or *exponentially* stable formation turns out to present a steady-state error in reality. In the case when this error can be reducible at will by a convenient tuning of some gains, this stability property is referred to as *practical*. Practical stability has been treated in several papers, see Chaillet and Loría (2006b), Chaillet and Loría (2008) and references therein. We will here give a very simple introductory example:

Example 2.1 *Consider the scalar system*

$$\dot{x} = -\theta x + d \tag{2.1}$$

where θ is a constant parameter and $d = d(t)$ is a non vanishing, time-varying disturbance. In this case, for any θ, the solutions are bounded by

$$|x(t)| \leq (|x(0) - \frac{\beta_d}{\theta}|)e^{-\theta t} + \frac{\beta_d}{\theta} \tag{2.2}$$

where $\beta_d = \sup_t d(t)$. We see that for any θ such that $\theta > \beta_d \delta$, the solutions converge exponentially to a ball around the origin of radius $\delta = \beta_d/\theta$.

Tools for a formal analysis of more involved parameterized time-varying systems will be given in Section 2.2. We will stress that ultimate boundedness as defined in Khalil (2002) is a *weaker* property than practical stability. For a system possessing the latter property, the vicinity of the origin to which the solutions converge may be made arbitrary small by convenient tuning of some parameters of the system, typically the control gains.

2.2 Definitions

Semiglobal and practical exponential stability properties pertain to parameterized nonlinear time-varying systems of the form

$$\dot{x} = f(t, x, \theta)\,, \tag{2.3}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$, $\theta \in \mathbb{R}^m$ is a vector of constant parameters and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x and piecewise continuous in t for any θ under consideration. θ is a free tuning parameter, that can for instance be a control gain, see Chaillet and Loría (2008) for details.

Definition 2.1 (UGPES) *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (2.3) is said to be* uniformly globally practically exponentially stable on Θ *if, given any $\delta > 0$, there exists a parameter $\theta^\star(\delta) \in \Theta$, and positive constants $k(\delta)$ and $\gamma(\delta)$ such that, for any $x_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}_{\geq 0}$ the solution of (2.3) satisfies, for all $t \geq t_0$,*

$$|x(t, t_0, x_0, \theta^\star)| \leq \delta + k(\delta)\,|x_0|\,e^{-\gamma(\delta)(t-t_0)}. \tag{2.4}$$

Definition 2.2 (USES) *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (2.3) is said to be* uniformly semiglobally exponentially stable on Θ *if, given any $\Delta > 0$, there exists a parameter $\theta^\star(\Delta) \in \Theta$ and positive constants $k(\Delta)$ and $\gamma(\Delta)$ such that, for any $x_0 \in \overline{\mathcal{B}}_\Delta$ and any $t_0 \in \mathbb{R}_{\geq 0}$ the solution of (2.3) satisfies, for all $t \geq t_0$,*

$$|x(t, t_0, x_0, \theta^\star)| \leq k(\Delta)\,|x_0|\,e^{-\gamma(\Delta)(t-t_0)}$$

Definition 2.3 (USPES) *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (2.3) is said to be* uniformly semiglobally practically exponentially stable on Θ *if, given any $\Delta > \delta > 0$, there exists a parameter $\theta^\star(\delta, \Delta) \in \Theta$ and positive constants $k(\delta, \Delta)$ and $\gamma(\delta, \Delta)$ such that, for any $x_0 \in \overline{\mathcal{B}}_\Delta$ and any $t_0 \in \mathbb{R}_{\geq 0}$ the solution of (2.3) satisfies, for all $t \geq t_0$,*

$$|x(t, t_0, x_0, \theta^\star)| \leq \delta + k(\delta, \Delta)\,|x_0|\,e^{-\gamma(\delta,\Delta)(t-t_0)}\,.$$

These properties are strongly related to their *asymptotic* counterpart (UGPAS, USAS and USPAS) defined and commented in detail in Chaillet and Loría (2006c, 2008). They are however stronger properties as they impose an *exponential* behavior of the solutions in the considered domain of the state-space and a linear dependency in the initial condition.

Remark 2.1 *The term* uniform *in the above definition is due to the requirement that the constants k and γ are independent of initial conditions. For time-varying systems the uniformity property is crucial as it provides certain robustness properties with respect to external disturbances. As pointed out in e.g. Loría and Panteley (2005), nonlinear time varying systems which are locally Lipschitz in t, and which are ULAS or ULES, are also locally input-to-state stable. On the contrary, systems without this property are not robust. An example is given in (Loría and Panteley, 2006, Proposition 6.1), of a system which solutions are exponentially convergent, but where the convergence rate depends on initial times. It is shown that it is possible to construct non-vanishing perturbations that destabilizes the system.*

Remark 2.2 *Note the difference of UGPES in Definition 2.1 and the definition of λ-UGPES in (Loría and Panteley, 2002, Definition 1). Although they also consider a parameterized nonlinear system, and the constants k and γ in Definition 2.1 may depend on a parameter λ, the stability is with respect to the origin (and not to a ball of radius δ).*

Remark 2.3 *We would also like to make the reader aware of the difference from the definition of UGPES (of impulsive systems) in (Dlala and Hammami, 2007, Definition 4.1), where the constants k and γ are independent of δ. Our approach, allows for the use of a Lyapunov function that depends on the parameter θ, (which again depends on δ).*

Remark 2.4 *Global practical uniform exponential stability was also defined in (Benabdallah et al., 2009, Definition 5), but with a different meaning than in 2.1, since the stability in the cited reference is with respect to a fixed set, and not a set that can be decreased by parameter tuning.*

In Chapter 5 we will deal with the attitude tracking of spacecraft. Since the Euler parameters, introduced in Section 1.3.2, will be used to describe the orientation error, any global/semiglobal results in the above setting are ruled out. This is due to the fact that the Euler parameters naturally entails two equilibrium points. We will therefore also need the following definition:

Definition 2.4 (UPES) *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system (2.3) is said to be* uniformly (locally) practically exponentially stable on Θ *if there exists $\Delta > 0$, and given any $\delta > 0$, there exists a parameter $\theta^\star(\delta, \Delta) \in \Theta$ and positive constants $k(\delta, \Delta)$ and $\gamma(\delta, \Delta)$ such that, for any $x_0 \in \overline{\mathcal{B}}_\Delta$ and any $t_0 \in \mathbb{R}_{\geq 0}$ the solution of (2.3) satisfies, for all $t \geq t_0$,*

$$|x(t, t_0, x_0, \theta^\star)| \leq \delta + k(\delta, \Delta)\, |x_0|\, e^{-\gamma(\delta,\Delta)(t-t_0)}\,.$$

2.3 Lyapunov sufficient conditions

We here present sufficient conditions for the above properties to hold. They are expressed as a condition on the sign of a Lyapunov-like function's derivative, on a restricted region of the state space.

2.3.1 UGPES

Theorem 2.1 (Sufficient condition for UGPES) *Let Θ be a subset of $\mathbb{R}^m$ and suppose that, given any $\delta > 0$, there exist a parameter $\theta^\star(\delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and positive constants $\kappa(\delta)$, $\underline{\kappa}(\delta)$, $\overline{\kappa}(\delta)$ such that, for all $x \in \mathbb{R}^n \setminus \overline{\mathcal{B}}_\delta$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\kappa}(\delta)\, |x|^p \leq V_\delta(t, x) \leq \overline{\kappa}(\delta)\, |x|^p\,, \tag{2.5}$$

$$\frac{\partial V_\delta}{\partial t}(t, x) + \frac{\partial V_\delta}{\partial x}(t, x) f(t, x, \theta^\star) \leq -\kappa(\delta)\, |x|^p\ , \tag{2.6}$$

where p denotes a positive constant. Then, under the condition that

$$\lim_{\delta \to 0} \frac{\overline{\kappa}(\delta)\delta^p}{\underline{\kappa}(\delta)} = 0\,, \tag{2.7}$$

the system $\dot{x} = f(t, x, \theta)$ introduced in (2.3) is UGPES on the parameter set Θ.

Proof. Let (2.5) and (2.6) hold for all $x \in \mathbb{R}^n \setminus \overline{\mathcal{B}}_{\tilde{\delta}}$ and all $t \in \mathbb{R}_{\geq 0}$. Along the solutions of (2.3), we get from (2.5) and (2.6) that

$$\begin{gathered} |x(t, t_0, x_0, \theta^\star)| > \tilde{\delta} \Rightarrow \\ \dot{V}_{\tilde{\delta}}(t, x(t, t_0, x_0, \theta^\star)) \leq -\kappa'(\tilde{\delta}) V_{\tilde{\delta}}(t, x(t, t_0, x_0, \theta^\star))\,, \end{gathered}$$

where $\kappa'(\tilde{\delta}) := \kappa(\tilde{\delta})/\overline{\kappa}(\tilde{\delta})$. Invoking (Chaillet and Loría, 2006c, Lemma 13), we then get that, for all $x_0 \in \mathbb{R}^n$, all $t_0 \in \mathbb{R}_{\geq 0}$ and all $t \geq t_0$,

$$|x(t, t_0, x_0, \theta^\star)| \leq \left(\frac{\overline{\kappa}(\tilde{\delta})\tilde{\delta}^p}{\underline{\kappa}(\tilde{\delta})}\right)^{1/p} + \left(\frac{\overline{\kappa}(\tilde{\delta})}{\underline{\kappa}(\tilde{\delta})}\right)^{1/p} |x_0|\, e^{-\kappa'(\tilde{\delta})(t-t_0)/p}.$$

In view of (2.7), we see that the quantity $\overline{\kappa}(\tilde{\delta})\tilde{\delta}^p/\underline{\kappa}(\tilde{\delta})$ may be reduced at will by choosing $\tilde{\delta}$ small enough. Therefore, (2.4) is satisfied with

$$\delta = \left(\frac{\overline{\kappa}(\tilde{\delta})\tilde{\delta}^p}{\underline{\kappa}(\tilde{\delta})}\right)^{1/p}, \quad k(\delta) = \left(\frac{\overline{\kappa}(\tilde{\delta})}{\underline{\kappa}(\tilde{\delta})}\right)^{1/p} \quad \text{and } \gamma\,(\delta) = \frac{\kappa'(\tilde{\delta})}{p},$$

and the conclusion follows. ■

Compared to classical results for Lyapunov stability, conditions (2.5) and (2.6) are natural (see (Khalil, 2002, Theorem 4.10)). For perturbed systems, (2.5) is notably satisfied by the Lyapunov function associated to the UGES of the origin of the corresponding nominal systems. (2.6) is similar to the Lyapunov sufficient condition for global ultimate boundedness (*cf. e.g.* Khalil (2002)). Intuitively, one may expect that these two requirements, when valid for any arbitrarily small δ, suffice to conclude UGPES. However, we see that an additional assumption (2.7) is required, establishing a relationship between the bounds on the Lyapunov function. Indeed, in the present framework, the Lyapunov function may here depend on the tuning parameter θ, and consequently on the radius δ. As clearly shown in Kokotovič and Marino (1986); Sepulchre (2000), this parametrization of the Lyapunov function may induce unexpected behaviors if (2.7) is not assumed.

2.3.2 USES

Theorem 2.2 (Sufficient condition for USES) *Let Θ be a subset of $\mathbb{R}^m$ and suppose that, given any $\Delta > 0$, there exist a parameter $\theta^\star(\Delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\Delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and positive constants $\kappa(\Delta)$, $\underline{\kappa}(\Delta)$, $\overline{\kappa}(\Delta)$ such that, for all $x \in \overline{\mathcal{B}}_\Delta$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\kappa}(\Delta)\,|x|^p \leq V_\Delta(t,x) \leq \overline{\kappa}(\Delta)\,|x|^p \tag{2.8}$$

$$\frac{\partial V_\Delta}{\partial t}(t,x) + \frac{\partial V_\Delta}{\partial x}(t,x) f(t,x,\theta^\star) \leq -\kappa(\Delta)\,|x|^p\ , \tag{2.9}$$

where p denotes a positive constant. Then, under the condition that

$$\lim_{\Delta\to\infty} \frac{\underline{\kappa}(\Delta)\Delta^p}{\overline{\kappa}(\Delta)} = \infty\,, \tag{2.10}$$

the system $\dot{x} = f(t,x,\theta)$ introduced in (2.3) is USES on the parameter set Θ.

The proof is omitted, but follows along the same lines as in the proof of Theorem 2.1 and Theorem 2.3.

2.3.3 USPES

Theorem 2.3 (Sufficient condition for USPES) *Let Θ be a subset of $\mathbb{R}^m$ and suppose that, given any $\Delta > \delta > 0$, there exist a parameter $\theta^\star(\delta, \Delta) \in \Theta$, a continuously differentiable Lyapunov function $V_{\delta,\Delta} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and positive constants $\kappa(\delta, \Delta)$, $\underline{\kappa}(\delta, \Delta)$, $\overline{\kappa}(\delta, \Delta)$ such that, for all $x \in \overline{\mathcal{B}}_\Delta \setminus \overline{\mathcal{B}}_\delta$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\kappa}(\delta, \Delta) \, |x|^p \leq V_{\delta,\Delta}(t, x) \leq \overline{\kappa}(\delta, \Delta) \, |x|^p \tag{2.11}$$

$$\frac{\partial V_{\delta,\Delta}}{\partial t}(t, x) + \frac{\partial V_{\delta,\Delta}}{\partial x}(t, x) f(t, x, \theta^\star) \leq -\kappa(\delta, \Delta) \, |x|^p \ , \tag{2.12}$$

where p denotes a positive constant. Assume also that, given any $\Delta^\star > \delta^\star > 0$, there exist $\Delta > \delta > 0$ such that

$$\frac{\overline{\kappa}(\delta, \Delta)\delta^p}{\underline{\kappa}(\delta, \Delta)} \leq \delta^\star \quad \textit{and} \quad \frac{\underline{\kappa}(\delta, \Delta)\Delta^p}{\overline{\kappa}(\delta, \Delta)} \geq \Delta^\star \, . \tag{2.13}$$

Then the system $\dot{x} = f(t, x, \theta)$ introduced in (2.3) is USPES on the parameter set Θ.

Proof. Let (2.11) and (2.12) hold for all $x \in \overline{\mathcal{B}}_{\tilde{\Delta}} \setminus \overline{\mathcal{B}}_{\tilde{\delta}}$ and all $t \in \mathbb{R}_{\geq 0}$. Let $\tilde{\Delta}$ be any positive constant and pick δ such that

$$\left(\frac{\overline{\kappa}\left(\tilde{\delta}, \tilde{\Delta}\right) \delta^p}{\underline{\kappa}\left(\tilde{\delta}, \tilde{\Delta}\right)} \right)^{\frac{1}{p}} < \tilde{\Delta},$$

which is always possible due to (2.13). This allows us to apply (Chaillet, 2006, Proposition 2.13), and we find that

$$|x_0| \leq \bar{\Delta} \implies |x(t, t_0, x_0, \theta^\star)| \leq \tilde{\Delta}, \quad \forall t \geq t_0,$$

where

$$\bar{\Delta} := \left(\frac{\underline{\kappa}\,(\delta, \Delta)\, \Delta^p}{\overline{\kappa}\,(\delta, \Delta)} \right)^{\frac{1}{p}} .$$

Note that solutions starting in $\overline{\mathcal{B}}_{\bar{\Delta}}$, will never escape $\overline{\mathcal{B}}_{\tilde{\Delta}}$, and in view of (Chaillet, 2006, Lemma 2.7), we have that for any $x_0 \in \overline{\mathcal{B}}_{\bar{\Delta}}$ and any $t_0 \in \mathbb{R}_{\geq 0}$,

$$|x\,(t, t_0, x_0, \theta^\star)| \leq \bar{\delta} + \left(\frac{\overline{\kappa}(\tilde{\delta}, \tilde{\Delta})}{\underline{\kappa}(\tilde{\delta}, \tilde{\Delta})} \right)^{1/p} |x_0| \, e^{-\kappa'(\tilde{\delta}, \tilde{\Delta})(t - t_0)/p},$$

where $\kappa'(\tilde{\delta},\tilde{\Delta}) := \kappa(\tilde{\delta},\tilde{\Delta})/\overline{\kappa}(\tilde{\delta},\tilde{\Delta})$, and

$$\bar{\delta} := \left(\frac{\overline{\kappa}(\tilde{\delta},\tilde{\Delta})\tilde{\delta}^p}{\underline{\kappa}(\tilde{\delta},\tilde{\Delta})}\right)^{1/p}.$$

In view of (2.13), we see that the quantity $\bar{\delta}$ may be reduced at will by choosing $\tilde{\delta}$ small enough, and that $\bar{\Delta}$ may be enlarged at will by choosing $\tilde{\Delta}$ large enough. Therefore, (2.3) is satisfied with

$$\delta = \bar{\delta}, \quad k(\delta,\Delta) = \left(\frac{\overline{\kappa}(\tilde{\delta},\tilde{\Delta})}{\underline{\kappa}(\tilde{\delta},\tilde{\Delta})}\right)^{1/p} \quad \text{and } \gamma\,(\delta,\Delta) = \frac{\kappa'(\tilde{\delta},\tilde{\Delta})}{p}$$

and the conclusion follows. ■

2.3.4 UPES

Theorem 2.4 (Sufficient condition for UPES) *Let Θ be a subset of $\mathbb{R}^m$ and suppose that, there exists $\Delta > 0$, and given any $\Delta > \delta > 0$, there exist a parameter $\theta^\star(\delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\delta : \mathbb{R}_{\geq 0} \times \overline{\mathcal{B}}_\Delta \to \mathbb{R}_{\geq 0}$ and positive constants $\kappa(\delta)$, $\underline{\kappa}(\delta)$, $\overline{\kappa}(\delta)$ such that, for all $x \in \overline{\mathcal{B}}_\Delta \setminus \overline{\mathcal{B}}_\delta$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\kappa}(\delta)\,|x|^p \leq V_\delta(t,x) \leq \overline{\kappa}(\delta)\,|x|^p \tag{2.14}$$

$$\frac{\partial V_\delta}{\partial t}(t,x) + \frac{\partial V_\delta}{\partial x}(t,x)f(t,x,\theta^\star) \leq -\kappa(\delta)\,|x|^p\ , \tag{2.15}$$

where p denotes a positive constant. Then, under the condition that

$$\lim_{\delta\to 0} \frac{\overline{\kappa}(\delta)\delta^p}{\underline{\kappa}(\delta)} = 0\,, \tag{2.16}$$

the system $\dot{x} = f(t,x,\theta)$ introduced in (2.3) is UPES on the parameter set Θ.

The proof is omitted, but follows along the same lines as Theorem 2.1.

2.3.5 *Practical* $\mathcal{K}$-exponential stability

For the sake of completeness, we will here briefly discuss the relations of the above stated definitions to $\mathcal{K}$-exponential stability. *Exponential stability in* *any ball of initial conditions* and *$\mathcal{K}$-exponential stability* - two equivalent definitions, see (Børhaug, 2008, Remark 2.1) - were defined in Sastry and Bodson (1994) and Sørdalen and Egeland (1995), respectively. These definitions are commonly applied to stability analysis of nonholonomic and underactuated systems. The following remark shows that *practical $\mathcal{K}$-exponential* and *practical asymptotic stability* are the same.

Remark 2.5 *The notion of* uniform practical $\mathcal{K}$-exponential stability*, i.e. for any $\delta > 0$, there exist $\theta^\star \in \Theta$, a function $\kappa_\delta \in \mathcal{K}$ and a positive constant $k_\delta > 0$, such that for all $x_0 \in \mathbb{R}^n$ and all $t_0 \geq 0$,*

$$|x(t, t_0, x_0, \theta^\star)| \leq \kappa_\delta(|x_0|) e^{-k_\delta(t-t_0)} + \delta, \quad \forall t \geq t_0, \tag{2.17}$$

and the notion of uniform practical asymptotic stability*, i.e. for any $\delta > 0$, there exist a function $\beta_\delta \in \mathcal{KL}$ and $\theta^\star \in \Theta$ such that for all $x_0 \in \mathbb{R}^n$, an all $t_0 \geq 0$,*

$$|x(t, t_0, x_0, \theta^\star)| \leq \beta_\delta(|x_0|, t - t_0) + \delta, \quad \forall t \geq t_0, \tag{2.18}$$

are equivalent.

Proof. The implication from (2.17) to (2.18) is trivial. For the implication in the opposite direction, consider the following: for all $x_0 \in \mathbb{R}^n$ and all $\delta > 0$, there exists a $T(|x_0|) \geq t_0$ such that $\beta_\delta(|x_0|, T_\delta) \leq \delta$. Then

$$|x(t)| \leq 2\delta, \quad \forall t \geq T_\delta(|x_0|).$$

By (Sontag, 1998, Lemma 8) we have that since $\beta_\delta \in \mathcal{KL}$, there exist $\alpha_{1_\delta}, \alpha_{2_\delta} \in \mathcal{K}_\infty$ such that for all $s \geq 0$, and for all $t \geq t_0 \geq 0$,

$$\beta_\delta(s, t - t_0) \leq \alpha_{1_\delta}(s)\, \alpha_{2_\delta}\left(e^{-(t-t_0)}\right).$$

Furthermore, for all $s \geq 0$ and all $t \leq T_\delta(s)$,

$$\begin{aligned} \beta_\delta(s, t) &\leq \alpha_{1_\delta}(s)\, \alpha_{2_\delta}(1)\, e^{T_\delta(s)-(t-t_0)} \\ &\leq \alpha_{1_\delta}(s)\, e^{T_\delta(s)} \alpha_{2_\delta}(1)\, e^{-(t-t_0)}. \end{aligned}$$

Let

$$\kappa_\delta(s) := \alpha_{1_\delta}(s)\, e^{T_\delta(s)} \alpha_{2_\delta}(1).$$

Then,

$$\begin{cases} |x(t)| \leq 2\delta & \forall t \geq T_\delta(|x_0|) \\ |x(t)| \leq \kappa_\delta(|x_0|)\, e^{-k_\delta(t-t_0)} + \delta & \forall t < T_\delta(|x_0|) \end{cases}$$

Let $\tilde{\delta} := 2\delta$. Then for all $t \geq t_0$,

$$|x(t)| \leq \kappa_\delta(|x_0|)\, e^{-k_\delta(t-t_0)} + \tilde{\delta}$$

which concludes the proof. ■

2.4 Stability of cascades

In this section we consider systems on the following cascaded structure:

$$\dot{x}_1 = f_1(t, x_1, \theta_1) + g(t, x, \theta) \tag{2.19}$$

$$\dot{x}_2 = f_2(t, x_2, \theta_2) \tag{2.20}$$

where $x := \left(x_1^\top, x_2^\top\right)^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $t \in \mathbb{R}_{\geq 0}$, $\theta := \left(\theta_1^\top, \theta_2^\top\right)^\top \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, f_1 and f_2 are locally Lipschitz in x and piecewise continuous in t for all θ under consideration. Stability of cascades for nonlinear nonautonomous systems has been thoroughly treated in the literature. Sufficient conditions for UGES can be found in Corless and Glielmo (1998) and Panteley et al. (1998); for UGAS can be found in Panteley and Loría (1998), Panteley and Loría (2001), Loría and Panteley (2005) and Tjønnås et al. (2006), and for UGPAS can be found in Chaillet and Loría (2006a) and Chaillet and Loría (2008). The above references contain general results for stability of cascades, where as the results of this section is mainly intended for the applications of this thesis.

2.4.1 UGPES

Theorem 2.5 *Under Assumption 2.1, 2.2 and 2.3, the system (2.19-2.20) is UGPES on $\Theta_1 \times \Theta_2$.*

Assumption 2.1 *Given any $\delta_1 > 0$, there exist a parameter $\theta_1^\star(\delta_1) \in \Theta_1$, a continuously differentiable Lyapunov function $V_{\delta_1} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ and positive constants $\underline{\kappa}\left(|\theta_1^\star|\right)$, $\overline{\kappa}\left(|\theta_1^\star|\right)$ (affine in $|\theta_1^\star|$) and an arbitrarily large $\kappa\left(|\theta_1^\star|\right)$ (affine in $|\theta_1^\star|$) such that, for all $x_1 \in \mathbb{R}^{n_1}$, and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\kappa}_1\left(|\theta_1^\star|\right)|x_1|^2 \leq V_{\delta_1}(t, x_1) \leq \overline{\kappa}_1\left(|\theta_1^\star|\right)|x_1|^2 \tag{2.21}$$

$$\frac{\partial V_{\delta_1}}{\partial t}(t, x_1) + \frac{\partial V_{\delta_1}}{\partial x_1}(t, x_1) f_1(t, x_1, \theta_1^\star) \leq -\kappa_1\left(|\theta_1^\star|\right)|x_1|^2 ,$$

Assumption 2.2 *There exist a positive constant c_2, and given any $\delta_2 > 0$, there exist a parameter $\theta_2^\star(\delta_2) \in \Theta_2$, a continuously differentiable Lyapunov function $V_{\delta_2} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ and positive constants $\underline{\kappa}_2\left(|\theta_2^\star|\right)$, $\overline{\kappa}_2\left(|\theta_2^\star|\right)$ (affine in $|\theta_2^\star|$) and an arbitrarily large $\kappa_2\left(|\theta_2^\star|\right)$ (affine in $|\theta_2^\star|$), such that, for all $x_2 \in \mathbb{R}^{n_2}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\kappa}_2\left(|\theta_2^\star|\right)|x_2|^2 \leq V_{\delta_2}\left(t, x_2\right) \leq \overline{\kappa}_2\left(|\theta_2^\star|\right)|x_2|^2 \tag{2.22}$$

$$\frac{\partial V_{\delta_2}}{\partial t}(t, x_2) + \frac{\partial V_{\delta_2}}{\partial x_2}(t, x_2) f_2(t, x_2, \theta_2^\star) \leq -\kappa_2(|\theta_2^\star|)|x_2|^2 + c_2|x_2|.$$

Assumption 2.3 *There exists a positive constant, $\tilde{c}$, such that the gradient of V_{δ_1} from Assumption 2.1 along the interconnection term for all $x = (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $t \in \mathbb{R}_{\geq 0}$, satisfies*

$$\frac{\partial V_{\delta_1}}{\partial x_1}(t, x_1) g(t, x, \theta_2^\star) \leq \tilde{c}|x_1| \sigma(x_1, x_2, \theta_2^\star)$$

where

$$\sigma(x_1, x_2, \theta_2^\star) \leq 1 + |\theta_2^\star||x_1| + (1 + |\theta_2^\star|)|x_2|.$$

The proof of Theorem 2.5 can be found in Appendix A.

2.4.2 UPES

We here state a local version of Theorem 2.5, a useful tool for stability analysis of attitude control using Euler parameters.

Theorem 2.6 *Under Assumption 2.4, 2.5 and 2.6, with*

$$\Delta := \min\{\Delta_1, \Delta_2\} > \delta := \max\{\delta_1, \delta_2\},$$

the system (2.19-2.20) is UPES on $\Theta_1 \times \Theta_2$.

Assumption 2.4 *Given any Δ_1 and any δ_1 such that $\Delta_1 > \delta_1 > 0$, there exist a parameter $\theta_1^\star(\delta_1) \in \Theta_1$, a continuously differentiable Lyapunov function $V_{\delta_1} : \mathbb{R}_{\geq 0} \times \overline{\mathcal{B}}_{\Delta_1} \to \mathbb{R}_{\geq 0}$ and positive constants $\underline{\kappa}_1(|\theta_1^\star|)$, $\overline{\kappa}_1(|\theta_1^\star|)$ (affine in $|\theta_1^\star|$) and an arbitrarily large $\kappa_1(|\theta_1^\star|)$ (affine in $|\theta_1^\star|$) such that, for all $x_1 \in \overline{\mathcal{B}}_{\Delta_1}$, and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\kappa}_1(|\theta_1^\star|)|x_1|^2 \leq V_{\delta_1}(t, x_1) \leq \overline{\kappa}_1(|\theta_1^\star|)|x_1|^2 \tag{2.23}$$

$$\frac{\partial V_{\delta_1}}{\partial t}(t, x_1) + \frac{\partial V_{\delta_1}}{\partial x_1}(t, x_1) f_1(t, x_1, \theta_1^\star) \leq -\kappa_1(|\theta_1^\star|)|x_1|^2, \tag{2.24}$$

Assumption 2.5 *There exist a positive constant c_2, and given any Δ_2 and any δ_2 such that $\Delta_2 > \delta_2 > 0$, there exist a parameter $\theta_2^\star(\delta_2) \in \Theta_2$, a continuously differentiable Lyapunov function $V_{\delta_2} : \mathbb{R}_{\geq 0} \times \overline{\mathcal{B}}_{\Delta_2} \to \mathbb{R}_{\geq 0}$ and positive constants $\underline{\kappa}_2(|\theta_2^\star|)$, $\overline{\kappa}_2(|\theta_2^\star|)$ (affine in $|\theta_2^\star|$) and an arbitrarily large $\kappa_2(|\theta_2^\star|)$ (affine in $|\theta_2^\star|$) such that, for all $x_2 \in \overline{\mathcal{B}}_{\Delta_2}$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\kappa}_2(|\theta_2^\star|)|x_2|^2 \leq V_{\delta_2}(t, x_2) \leq \overline{\kappa}_2(|\theta_2^\star|)|x_2|^2 \tag{2.25}$$

$$\frac{\partial V_{\delta_2}}{\partial t}(t, x_2) + \frac{\partial V_{\delta_2}}{\partial x_2}(t, x_2) f_2(t, x_2, \theta_2^\star) \leq -\kappa_2(|\theta_2^\star|)|x_2|^2 + c_2|x_2|.$$

Assumption 2.6 *There exists a positive constant, $\tilde{c}$, such that the gradient of V_{δ_1} from Assumption 2.1 along the interconnection term for all $x = \left(x_1^\top, x_2^\top\right)^\top \in \overline{\mathcal{B}}_{\Delta_1} \times \overline{\mathcal{B}}_{\Delta_2}$ and all $t \in \mathbb{R}_{\geq 0}$, satisfies*

$$\frac{\partial V_{\delta_1}}{\partial x_1}(t, x_1)\, g(t, x, \theta_2^\star) \leq \tilde{c}\, |x_1|\, \sigma(x_1, x_2, \theta_2^\star)$$

where

$$\sigma(x_1, x_2, \theta_2^\star) \leq 1 + |\theta_2^\star|\, |x_1| + (1 + |\theta_2^\star|)\, |x_2| \, .$$

The proof follows along the lines of the proof of Theorem 2.5, which is given in Appendix A.

2.4.3 UGPAS

The next theorem shows that we can relax the conditions on the interconnection term, at the price of only achieving UGPAS instead of UGPES. More general sufficient conditions for a system of the structure (2.20-2.20) to be UGPAS, have already been given in Chaillet and Loría (2006b). However, for certain systems on a cascaded structure, such as the dynamics of a leader follower formation presented in the subsequent chapters, the conditions on the interconnection term might not be fulfilled. The main reason it that the trajectory based proof technique in Chaillet and Loría (2006b), does not, in general, allow for the interconnection to depend on θ_2. It should be noted that there is a relaxation stated in (Chaillet and Loría, 2006b, Remark 2) that allows for the interconnection term to depend on θ_2, but this relaxation is not applicable for the leader-follower formation considered in this thesis. We therefore present a cascaded theorem, which proof is based on repeated use of Young's Inequality as opposed to the trajectory based proof technique in Chaillet and Loría (2006b). The implication is that the former allows for the interconnection term to depend on the tuning parameters of the driving subsystem, where as the latter allows it to depend on the tuning parameters of the driven subsystem.

Theorem 2.7 *Under Assumption 2.1 with $\underline{\kappa}_1$ independent of $|\theta_1^\star|$, Assumption 2.2 with $\underline{\kappa}_2$ independent of $|\theta_2^\star|$ and Assumption 2.7, the system (2.19-2.20) is UGPAS on $\Theta_1 \times \Theta_2$.*

Assumption 2.7 *There exists a positive constant, $\tilde{c}$, such that the gradient of V_{δ_1} from Assumption 2.1 along the interconnection term satisfies*

$$\frac{\partial V_{\delta_1}}{\partial x_1}(t, x_1)\, g(t, x, \theta_2^\star) \leq \tilde{c}\, |x_1|\, \sigma(x_1, x_2, \theta_2^\star)$$

where

$$\sigma(x_1, x_2, \theta_2^\star) \le 1 + |\theta_2^\star|\,|x_1| + (1 + |x_1| + |\theta_2^\star| + |x_1|\,|\theta_2^\star|) \sum_{q=1}^{Q} |x_2|^q \,.$$

The proof of Theorem 2.7 can be found in Appendix A.

2.5 Robustness of ISS systems with respect to a class of non-bounded energy inputs

2.5.1 Terminology

We next recall some classical definitions related to the stability and robustness of nonlinear systems of the form

$$\dot{x} = f(x, u), \tag{2.26}$$

where $x \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is locally Lipschitz in x. The class $\mathcal{U}$ of external inputs u that we consider consists of a subset of all signals $u : \mathbb{R}_{\ge 0} \to \mathbb{R}^p$ that are measurable and locally essentially bounded.

Definition 2.5 *Let δ be a positive number and u be a given signal in $\mathcal{U}$. The ball $\overline{\mathcal{B}}_\delta$ is said to be globally asymptotically stable (GAS) for (2.26) if there exists a class $\mathcal{KL}$ function β such that the solution of (2.26) from any initial state $x_0 \in \mathbb{R}^n$ satisfies*

$$|x(t, x_0, u)| \le \delta + \beta(|x_0|, t), \quad \forall t \ge 0. \tag{2.27}$$

Definition 2.6 *The ball $\overline{\mathcal{B}}_\delta$ is said to be globally exponentially stable (GES) for (2.26) if Definition 2.5 holds with $\beta(r, s) = kre^{-\gamma s}$ for some positive constants k and γ.*

Definition 2.7 *The system $\dot{x} = f(x, u)$ is said to be input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$, the solution of (2.26) satisfies*

$$|x(t, x_0, u)| \le \beta(|x_0|, t) + \gamma(||u(s)||_\infty)\,, \quad \forall t \ge 0\,.$$

ISS thus imposes an asymptotic decay of the norm of the state up to a function of the *amplitude* $||u||_\infty$ of the input signal. We also recall the following well-known Lyapunov characterization of ISS, originally established in Praly and Wang (1996) and thus extending the original characterization proposed in Sontag and Wang (1995).

Proposition 2.1 *The system (2.26) is ISS if and only if there exist* $\underline{\alpha}, \overline{\alpha}, \gamma \in \mathcal{K}_\infty$ *and* $\kappa > 0$ *such that, for all* $x \in \mathbb{R}^n$ *and all* $u \in \mathbb{R}^p$,

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)$$

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\kappa V(x) + \gamma(|u|)\,.$$

γ *is then called a* supply rate *for (2.26).*

Remark 2.6 *Since ISS implies iISS (cf. Sontag (1998)), it can be shown that the solutions of any ISS system with supply rate* γ *satisfies, for all* $x_0 \in \mathbb{R}^n$,

$$|x(t,x_0,u)| \leq \beta(|x_0|,t) + \eta\left(\int_0^t \gamma(|u(\tau)|)\mathrm{d}\tau\right)\,, \quad \forall t \geq 0\,,$$

where $\beta \in \mathcal{KL}$ *and* $\eta \in \mathcal{K}_\infty$.

The above remark establishes a link between a measure of the energy fed into the system and the norm of the state: if this energy is small, then the state will eventually be small. However, stated as above, the ISS property does not provide any information on the behavior of the system when this energy is not finite, that is if the perturbation persistently excites the system. In the same way, the statement of Definition 2.7 is not relevant for signals whose supremum is unbounded. Both these types of signals are relevant for control applications, and in particular for spacecraft formations. This motivates the introduction of the following class of signals.

Definition 2.8 *Given some constants* $E, T > 0$ *and and some function* $\gamma \in \mathcal{K}$, *the set* $\mathcal{W}_\gamma(E,T)$ *denotes the set of all signals* $u \in \mathcal{U}$ *satisfying, for all* $t \in \mathbb{R}_{\geq 0}$,

$$\int_t^{t+T} \gamma(|u(s)|)\mathrm{d}s \leq E\,.$$

Any signal u belonging to the class $\mathcal{W}_\gamma(E,T)$ has therefore a limited excitation *in average*. The main concern here is the measure E of the maximum energy that can be fed into the system over a moving time window of given length T. Signals of this class are not necessarily globally essentially bounded, nor are they required to have a finite energy, as illustrated by the following examples. Robustness to this class of signal thus constitutes an extension of typical properties of ISS systems.

Example 2.2 *1. Essentially bounded signals: given any* $T > 0$ *and any* $\gamma \in \mathcal{K}$*, if* $||u||_\infty$ *is finite then it holds that* $u \in \mathcal{W}_\gamma(T\gamma(||u||_\infty), T)$*. We stress that this includes signals with infinite energy (think for instance of constant non-zero signals).*

2. Unbounded signals: given any $T > 0$ *and any* $\gamma \in \mathcal{K}$*, the following signal belongs to* $\mathcal{W}_\gamma(1,T)$ *and satisfies* $\limsup_{t\to\infty} |u(t)| = +\infty$*:*

$$u(t) := \begin{cases} 2k & \textit{if} \quad t \in [2kT; 2kT + \frac{1}{2k}]\,,\ k \in \mathbb{N} \\ 0 & \textit{otherwise.} \end{cases}$$

2.5.2 Robustness of ISS systems w.r.t. signals in class $\mathcal{W}$

The main contribution of this article is the following result, which establishes that the impact of exogenous signals on the qualitative behavior of an ISS systems is negligible if the average excitation of this signal is sufficiently small.

Theorem 2.8 *Assume that the system* $\dot{x} = f(x,u)$ *is ISS. Then, there exists a class* $\mathcal{K}_\infty$ *function* γ *and a class* $\mathcal{KL}$ *function* β *and, given any precision* $\delta > 0$ *and any time window* $T > 0$*, there exists a positive average excitation* $E(T,\delta)$ *such that, given any* $u \in \mathcal{W}_\gamma(E,T)$*, the ball* $\overline{\mathcal{B}}_\delta$ *is GAS.*

The above result adds another brick in the wall of nice properties induced by ISS, cf. Sontag (2007) and references therein. It ensures that, provided that steady-state error δ can be tolerated, then every ISS system is robust to a class of disturbances with esuriently small average excitation. Of course, the greater imprecise δ one may tolerate, the larger the class admissible perturbations.

It is worth stressing that the $\mathcal{KL}$ estimate of the solutions is independent of the required precision δ. This implies, in particular, that the expected transient overshoot and decay rate are independent of the aimed precision.

Proof of Theorem 2.8. In view of (Praly and Wang, 1996, Lemma 11) and (Angeli et al., 2000, Remark 2.4), there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$ and γ, and a positive constant κ such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{2.28}$$

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\kappa V(x) + \gamma(|u|)\,. \tag{2.29}$$

Let $w(t) := V(x(t, x_0, u))$. Then it holds in view of (2.29) that

$$\begin{aligned} \dot{w}(t) &= \dot{V}(x(t,x_0,u)) \\ &\leq -\kappa V(x(t,x_0,u)) + \gamma(|u(t)|) \\ &\leq -\kappa w(t) + \gamma(|u(t)|)\,. \end{aligned}$$

In particular, it holds that, for all $t \geq 0$,

$$w(t) \leq w(0)e^{-\kappa t} + \int_0^t \gamma(|u(s)|)\mathrm{d}s\,. \tag{2.30}$$

Assuming that u belongs to the class $\mathcal{W}_\gamma(E,T)$, for some arbitrary constants $E, T > 0$, it follows that

$$w(T) \leq w(0)e^{-\kappa T} + \int_0^T \gamma(|u(s)|)\mathrm{d}s \leq w(0)e^{-\kappa T} + E\,.$$

Considering this inequality recursively, it follows that, for each $\ell \in \mathbb{N}_{\geq 1}$,

$$\begin{aligned} w(\ell T) &\leq w(0)e^{-\ell\kappa T} + E\sum_{j=0}^{k-1} e^{-j\kappa T} \\ &\leq w(0)e^{-\ell\kappa T} + E\sum_{j\geq 0} e^{-j\kappa T} \\ &\leq w(0)e^{-\ell\kappa T} + E\frac{e^{\kappa T}}{e^{\kappa T}-1}\,. \end{aligned} \tag{2.31}$$

Given any $t \geq 0$, pick ℓ as $\lfloor t/T \rfloor$ and define $t' := t - \ell T$. Note that $t' \in [0,T]$. Then, it follows from (2.30) that

$$w(t) \leq w(\ell T)e^{-\kappa t'} + \int_{\ell T}^t \gamma(|u(s)|)\mathrm{d}s \leq w(\ell T)e^{-\kappa t'} + E\,,$$

which, in view of (2.31), implies that

$$
\begin{aligned}
w(t) &\le \left(w(0)e^{-\ell\kappa T} + E\frac{e^{\kappa T}}{e^{\kappa T}-1}\right)e^{-t'} + E \\
&\le w(0)e^{-k(\ell T+t')} + E\left(1+\frac{e^{\kappa T}}{e^{\kappa T}-1}\right) \\
&\le w(0)e^{-\kappa t} + \frac{2e^{\kappa T}-1}{e^{\kappa T}-1}E\,.
\end{aligned}
$$

Recalling that $w(t) = V(x(t,x_0,u))$, it follows that

$$
V(x(t,x_0,u)) \le V(x_0)e^{-\kappa t} + \frac{2e^{\kappa T}-1}{e^{\kappa T}-1}E\,,
$$

which implies, in view of (2.28), that

$$
\underline{\alpha}(|x(t,x_0,u)|) \le \overline{\alpha}(|x_0|)e^{-\kappa t} + \frac{2e^{\kappa T}-1}{e^{\kappa T}-1}E\,,
$$

Recalling that $\underline{\alpha}^{-1}(a+b) \le \underline{\alpha}^{-1}(2a)+\underline{\alpha}^{-1}(2b)$ as $\underline{\alpha} \in \mathcal{K}_\infty$, we finally obtain that, given any $x_0 \in \mathbb{R}^n$, any $u \in \mathcal{W}_\gamma(E,T)$ and any $t \ge 0$,

$$
|x(t,x_0,u)| \le \underline{\alpha}^{-1}\left(2\overline{\alpha}(|x_0|)e^{-\kappa t}\right) + \underline{\alpha}^{-1}\left(2E\frac{2e^{\kappa T}-1}{e^{\kappa T}-1}\right). \tag{2.32}
$$

Given any $T, \delta \ge 0$, the following choice of E:

$$
E(T,\delta) \le \frac{\underline{\alpha}(\delta)}{2}\frac{e^{\kappa T}-1}{2e^{\kappa T}-1}\,. \tag{2.33}
$$

ensures that

$$
\underline{\alpha}^{-1}\left(2E\frac{2e^{\kappa T}-1}{e^{\kappa T}-1}\right) \le \delta
$$

and the conclusion follows in view of (2.32) with the $\mathcal{KL}$ function

$$
\beta(s,t) := \underline{\alpha}^{-1}\left(2\overline{\alpha}(s)e^{-\kappa t}\right)\,, \quad \forall s,t \ge 0\,.
$$

■

Known explicit Lyapunov function

It follows along the proof of Theorem 2.8 that an explicit bound on the tolerable average excitation can be provided if an ISS Lyapunov is known. More precisely, we state the following result.

Corollary 2.1 *Assume there exists a continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*, a positive constant* κ *and class* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}$ *and* $\overline{\alpha}$ *such that, for all* $x \in \mathbb{R}^n$ *and all* $u \in \mathbb{R}^m$*,*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{2.34}$$

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\kappa V(x) + \gamma(|u|)\,. \tag{2.35}$$

Given any precision $\delta > 0$ *and any time window* $T > 0$*, consider any average excitation satisfying*

$$E(T,\delta) \leq \frac{\underline{\alpha}(\delta)}{2} \frac{e^{\kappa T} - 1}{2e^{\kappa T} - 1}\,. \tag{2.36}$$

Then, for any $u \in \mathcal{W}_\gamma(E,T)$ *and any* $x_0 \in \mathbb{R}^n$*,* $\overline{\mathcal{B}}_\delta$ *is GAS for* $\dot{x} = f(x,u)$*.*

The above statements shows that, by knowing a Lyapunov function associated to the ISS of a system, and in particular its dissipation rate γ, one is able to explicitly identify the class $\mathcal{W}_\gamma(E,T)$ to which it is robust up to the arbitrary precision δ.

In a similar way, we can state sufficient condition for global exponential stability of some neighborhood of the origin. This result also trivially follows from the proof of Theorem 2.8.

Corollary 2.2 *If the conditions of Corollary 2.1 are satisfied with* $\underline{\alpha}(s) = \underline{c}s^p$ *and* $\overline{\alpha}(s) = \overline{c}s^p$*, with* $\underline{c}, \overline{c}, p$ *positive constants, then, given any* $T, \delta > 0$*, the ball* $\bar{\mathcal{B}}_\delta$ *is GES for (2.26) with any signal* $u \in \mathcal{W}_\gamma(E,T)$ *provided that*

$$E(T,\delta) \leq \frac{\underline{c}\delta^p}{2} \frac{e^{\kappa T} - 1}{2e^{\kappa T} - 1}\,.$$

Chapter 3

Modeling

This chapter is devoted to modeling of the formations of spacecraft. In Section 3.1 the model for the translational case is derived, and Section 3.2 contains the model for the rotational case.

3.1 Translational case

The model of the spacecraft can be derived with respect to the inertial reference frame, or with respect to a moving reference frame. The moving reference frame can either be an orbital reference frame of a prescribed motion or a reference frame attached to one of the spacecraft in the formation. Let the position of the spacecraft be described by the vector $\vec{r}$, let $\vec{r}_o$ be the vector describing the origin of the moving reference frame $\mathcal{F}_o$, and define $\vec{p} := \vec{r} - \vec{r}_o$. The acceleration is then given by the following equation, which is a result of applying the rules for differentiation in moving frames, see e.g. Egeland and Gravdahl (2002):

$$\frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r} = \frac{{}^o\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\omega}_{io} \times \vec{r}$$

$$\begin{aligned} \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r} &= \frac{{}^o\mathrm{d}}{\mathrm{d}t}\left(\frac{{}^o\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\omega}_{io} \times \vec{r}\right) + \vec{\omega}_{io} \times \left(\frac{{}^o\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\omega}_{io} \times \vec{r}\right) \\ &= \frac{{}^o\mathrm{d}^2}{\mathrm{d}t^2}\vec{r} + 2\vec{\omega}_{io} \times \frac{{}^o\mathrm{d}}{\mathrm{d}t}\vec{r} + \left(\frac{{}^o\mathrm{d}}{\mathrm{d}t}\vec{\omega}_{io}\right) \times \vec{r} + \vec{\omega}_{io} \times (\vec{\omega}_{io} \times \vec{r}) \quad (3.1) \end{aligned}$$

Here, $\vec{\omega}_{io}$ is the angular velocity of the moving reference frame, relative to the inertial frame. The unit vectors of $\mathcal{F}_o$ are chosen such that $\vec{o}_1$ points in the anti-nadir direction, $\vec{o}_3$ points in the direction of the orbit normal, and finally $\vec{o}_2$ completes the right-handed orthogonal frame. The rotation matrix between the orbital frame and the inertial frame is given by

$$R_o^i = \begin{bmatrix} o_1^i & o_2^i & o_3^i \end{bmatrix} \in SO\left(3\right),$$

where

$$o_1^i := \frac{r_o^i}{|r_o|}, \quad o_2^i := o_3^i \times o_2^i, \quad o_3^i := \frac{r_o^i \times \dot{r}_o^i}{|r_o \times \dot{r}_o|}.$$

The matrix can also be expressed in terms of orbital parameters, as three consecutive rotations:

$$R_o^i = R_z\left(\Omega_o\right) R_x\left(i_o\right) R_z\left(\tilde{\omega}_o + \nu_o\right),$$

where Ω_o is the longitude of the ascending node, i_o is the inclination, $\tilde{\omega}_o$ is the argument of perigee and ν_o is the true anomaly. The subscript o is to emphasize that these are the orbital parameters describing the motion of the point o. The angular velocity can be given as a coordinate vector in terms of the orbital parameters as:

$$\begin{aligned}
\omega_{io}^i &= \begin{bmatrix} 0 \\ 0 \\ \dot{\Omega}_o \end{bmatrix} + R_{z,\Omega_o} \begin{bmatrix} \dot{i}_o \\ 0 \\ 0 \end{bmatrix} + R_{z,\Omega_o} R_{x,i_o} \begin{bmatrix} 0 \\ 0 \\ \dot{\tilde{\omega}}_o + \dot{\nu}_o \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\Omega}_o \end{bmatrix} + \begin{bmatrix} \cos\Omega_o & -\sin\Omega_o & 0 \\ \sin\Omega_o & \cos\Omega_o & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{i}_o \\ 0 \\ 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \cos\Omega_o & -\cos i_o \sin\Omega_o & \sin i_o \sin\Omega_o \\ \sin\Omega_o & \cos i_o \cos\Omega_o & -\sin i_o \cos\Omega_o \\ 0 & \sin i_o & \cos i_o \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\tilde{\omega}}_o + \dot{\nu}_o \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\Omega}_o \end{bmatrix} + \begin{bmatrix} \dot{i}_o \cos\Omega_o \\ \dot{i}_o \sin\Omega_o \\ 0 \end{bmatrix} + \begin{bmatrix} \left(\dot{\tilde{\omega}}_o + \dot{\nu}_o\right) \sin i_o \sin\Omega_o \\ -\left(\dot{\tilde{\omega}}_o + \dot{\nu}_o\right) \sin i_o \cos\Omega_o \\ \left(\dot{\tilde{\omega}}_o + \dot{\nu}_o\right) \cos i_o \end{bmatrix} \\
&= T_\Theta^{-1}\left(\Theta\right) \dot{\Theta}
\end{aligned}$$

with $\Theta = \left(\Omega_o, i_o, \tilde{\omega}_o + \nu_o\right)$ and

$$T_\Theta^{-1} = \begin{bmatrix} 0 & \cos\Omega_o & \sin i_o \sin\Omega_o \\ 0 & \sin\Omega_o & -\sin i_o \cos\Omega_o \\ 1 & 0 & \cos i_o \end{bmatrix}.$$

Hence,

$$T_\Theta(\Theta) = \begin{bmatrix} -(\cos i_o)\frac{\sin\Omega_o}{\sin i_o} & (\cos i_o)\frac{\cos\Omega_o}{\sin i_o} & 1 \\ \cos\Omega_o & \sin\Omega_o & 0 \\ \frac{\sin\Omega_o}{\sin i_o} & -\frac{\cos\Omega_o}{\sin i_o} & 0 \end{bmatrix}.$$

Or, the angular velocity can be expressed in the leader spacecraft frame as:

$$\begin{aligned} \omega_{io}^o &= R_{z,-(\tilde{\omega}_o+\nu_o)}R_{x,-i_o}\begin{bmatrix}0\\0\\\dot{\Omega}_o\end{bmatrix} + R_{z,-(\tilde{\omega}_o+\nu_o)}\begin{bmatrix}\dot{i}_o\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot{\omega}_o+\dot{\nu}_o\end{bmatrix} \\ &= \begin{bmatrix} \cos(\nu_o+\tilde{\omega}_o) & \sin(\nu_o+\tilde{\omega}_o)\cos i_o & \sin(\nu_o+\tilde{\omega}_o)\sin i_o \\ -\sin(\nu_o+\tilde{\omega}_o) & \cos(\nu_o+\tilde{\omega}_o)\cos i_o & \cos(\nu_o+\tilde{\omega}_o)\sin i_o \\ 0 & -\sin i_o & \cos i_o \end{bmatrix}\begin{bmatrix}0\\0\\\dot{\Omega}_o\end{bmatrix} \\ &\quad + \begin{bmatrix} \cos(\tilde{\omega}_o+\nu_o) & \sin(\tilde{\omega}_o+\nu_o) & 0 \\ -\sin(\tilde{\omega}_o+\nu_o) & \cos(\tilde{\omega}_o+\nu_o) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}\dot{i}_o\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot{\tilde{\omega}}_o+\dot{\nu}_o\end{bmatrix} \\ &= \begin{bmatrix}\dot{\Omega}_o\sin(\nu+\tilde{\omega}_o)\sin i_o\\\dot{\Omega}_o\cos(\nu+\tilde{\omega}_o)\sin i_o\\\dot{\Omega}_o\cos i_o\end{bmatrix} + \begin{bmatrix}\dot{i}_o\cos(\tilde{\omega}_o+\nu_o)\\-\dot{i}_o\sin(\tilde{\omega}_o+\nu_o)\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot{\tilde{\omega}}_o+\dot{\nu}_o\end{bmatrix} \\ &= U_\Theta^{-1}\dot{\Theta} \end{aligned}$$

with

$$U_\Theta^{-1} = \begin{bmatrix} \sin(\nu+\tilde{\omega})\sin i & \cos(\tilde{\omega}+\nu) & 0 \\ \cos(\nu+\tilde{\omega})\sin i & -\sin(\tilde{\omega}+\nu) & 0 \\ \cos i & 0 & 1 \end{bmatrix}.$$

Notice that we have not chosen the origin of the moving reference frame yet. It can be chosen to satisfy any point of interest with respect to the spacecraft, and a suitable choice is to let the origin be a solution to the differential equation

$$\frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}_o = -\frac{\mu}{|\vec{r}_o|^3}\vec{r}_o \tag{3.2}$$

where μ is the gravitational parameter of Earth. By choosing the origin of the orbit frame to be a solution of (3.2), the orbital elements $\tilde{\omega}_o$, i_o and Ω_o are constant, and the angular velocity of the orbital frame becomes simply

$$\omega_{io}^o = (0, 0, \dot{\nu}_o)^\top. \tag{3.3}$$

From *Newton's Second Law*, we have that

$$m\frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r} = \vec{f}$$

where $\vec{f}$ is the resultant force acting on the center of gravity of the object with mass m. Expressing the model (3.1) in matrix form, we find that

$$\frac{f^o}{m} = (\ddot{r}_o^o + \ddot{p}^o) + 2S(\omega_{io}^o)(\dot{r}_o^o + \dot{p}^o) + (S^2(\omega_{io}^o) + S(\dot{\omega}_{io}^o))(r_o^o + p^o)$$

Remark 3.1 *An alternative approach is to differentiate the expression* $r^i = R_o^i r^o$, *twice. This, by using that* $\dot{R}_o^i = R_o^i S(\omega_{io}^o)$, *directly leads to*

$$\ddot{r}^i = R_o^i \left[\ddot{r}^o + 2S(\omega_{io}^o)\dot{r}^o + \left(S^2(\omega_{io}^o) + S(\dot{\omega}_{io}^o)\right)r^o\right]$$

Now, Newton's law states that $\ddot{r}^i = f^i/m$, *where* $f^i \in \mathbb{R}^3$ *consists of all the forces acting on the spacecraft and* m *is the spacecraft mass, and which equivalently can be stated as* $\ddot{r}^i = R_o^i f^o/m$. *Inserting this expression and using that* $r = r_o + p$, *gives*

$$\frac{f^o}{m} = (\ddot{r}_o^o + \ddot{p}^o) + 2S(\omega_{io}^o)(\dot{r}_o^o + \dot{p}^o) + (S^2(\omega_{io}^o) + S(\dot{\omega}_{io}^o))(r_o^o + p^o).$$

Now, consider that the motion of spacecraft j is given by these same equations, but we use the subscript j, to denote spacecraft j:

$$\frac{f_j^o}{m_j} = (\ddot{r}_o^o + \ddot{p}_j^o) + 2S(\omega_{io}^o)\left(\dot{r}_o^o + \dot{p}_j^o\right) + (S^2(\omega_{io}^o) + S(\dot{\omega}_{io}^o))\left(r_o^o + p_j^o\right). \quad (3.4)$$

This is the model given in Ploen et al. (2004b). Since

$$\ddot{r}_o^i = R_o^i((S^2(\omega_{io}^o) + S(\dot{\omega}_{io}^o))r_o^o + 2S(\omega_{io}^o)\dot{r}_o^o + \ddot{r}_o^o)$$

and, if we choose the origin of $\mathcal{F}_o$ to satisfy the coordinate version of (3.2),

$$\ddot{r}_o^i = -\frac{\mu}{|r_o|^3} r_o^i,$$

we get that

$$\frac{f_j^o}{m_j} = \ddot{p}_j^o + 2S(\omega_{io}^o)\dot{p}_j^o + (S^2(\omega_{io}^o) + S(\dot{\omega}_{io}^o))p_j^o - \frac{\mu}{|r_o|^3} r_o^o.$$

Furthermore, by using that f_j is composed of gravitational forces, control forces u_j and other forces d_j, the model can be written as:

$$m_j((S^2(\omega_{io}^o) + S(\dot{\omega}_{io}^o))p_j^o + 2S(\omega_{io}^o)\dot{p}_j^o + \ddot{p}_j^o)) + n(r_o, p_j) = u_j^o + d_j^o. \quad (3.5)$$

with

$$n(r_o, p_j) := m_j \mu \left(\frac{r_o^o + p_j^o}{\left|r_o^o + p_j^o\right|^3} - \frac{r_o^o}{|r_o|^3} \right).$$

Remark 3.2 *In Ploen et al. (2004b), each spacecraft j is considered as a part of what is called a virtual structure. A convenient model is found by stacking the position vector of each spacecraft, and letting the origin of the reference frame $\mathcal{F}_o$ for example be the center of mass of the formation.*

We will now derive the relative dynamics of spacecraft k and j. Define $p_{jk} := p_k - p_j$ as the relative position vector. By (3.4), the equation of motion of spacecraft k relative to spacecraft j is found to be:

$$m_k((S^2(\omega_{io}^o) + S(\dot{\omega}_{io}^o))p_{jk}^o + 2S(\omega_{io}^o)\dot{p}_{jk}^o + \ddot{p}_{jk}^o) = f_k^o - \frac{m_k}{m_j}f_k^o$$

Again, by (3.3), and by separating f_j and f_k into gravitational forces, control forces and other forces, the equation become:

$$\begin{aligned} u_k^o + d_k^o - \frac{m_k}{m_j}\left(u_j^o + d_j^o\right) &= m_k(S^2(\omega_{io}^o) + S(\dot{\omega}_{io}^o))p_{jk}^o + 2m_k S(\omega_{io}^o)\dot{p}_{jk}^o \\ &\quad + m_k\ddot{p}_{jk}^o + n(r_o, p_j, p_{jk}) \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} n(r_o, p_j, p_{jk}) &:= m_k\mu\left(\frac{r_o^o + p_k^o}{|r_o + p_k|^3} - \frac{r_o^o + p_j^o}{|r_o + p_j|^3}\right) \\ &= m_k\mu\left(\frac{r_o^o + p_{jk}^o + p_j^o}{|r_o + p_{jk} + p_j|^3} - \frac{r_o^o + p_j^o}{|r_o + p_j|^3}\right) \end{aligned}$$

For later reference we will now give a compact description of the models used in the chapters to follow.

3.1.1 Model with reference frame following Keplerian orbit

When the reference frame is moving in a Keplerian orbit, the model of the leader is given by (3.5), which can be rewritten as:

$$m_l\ddot{p} + C_l(\dot{\nu}_o)\dot{p} + D_l(\dot{\nu}_o, \ddot{\nu}_o)p + n_l(r_o, p) = u_l + d_l \tag{3.7}$$

where (borrowing notation from Kristiansen et al. (2006b))

$$C_l(\dot{\nu}_o) := 2m_l\dot{\nu}_o\bar{C}, \quad \bar{C} := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{SS}(3),$$

$$D_l(\dot{\nu}_o, \ddot{\nu}_o) := \quad m_l \dot{\nu}_o^2 \bar{D} + m_l \ddot{\nu}_o \bar{C},$$
$$\bar{D} := \quad \mathrm{diag}(-1, -1, 0) \in \mathbb{R}^{3\times 3},$$
$$n_l(r_o, p) := m_l \mu \left(\frac{r_o + p}{|r_o + p|^3} - \frac{r_o}{|r_o|^3} \right),$$

m_l is the mass of the leader spacecraft, and u_l and d_l are the control- and disturbance vectors, respectively, acting on the leader. The model describing the motion of the follower spacecraft relative to the leader is given by (3.6) and can be written as

$$m_f \ddot{\rho} + C_f(\dot{\nu}_o) \dot{\rho} + D_f(\dot{\nu}_o, \ddot{\nu}_o) \rho + n_f(r_o, p, \rho) = u_f + d_f - \frac{m_f}{m_l}(u_l + d_l), \quad (3.8)$$

with

$$D_f(\dot{\nu}_o, \ddot{\nu}_o) := m_f \dot{\nu}_o^2 \bar{D} + m_f \ddot{\nu}_o \bar{C},$$

and

$$n_f(r_o, p, \rho) := m_f \mu \left(\frac{r_o + p + \rho}{|r_o + \rho + p|^3} - \frac{r_o + p}{|r_o + p|^3} \right).$$

m_f is the mass of the follower spacecraft, where as u_f and d_f are the control- and disturbance vectors, respectively. All vectors, both for the leader and the follower spacecraft, are expressed in an orbital frame, with the origin satisfying Newton's gravitational law. The superscript to denote the reference frame o, has been dropped out of notational convenience and the subscripts of (3.5) and (3.6) have been replaced with l and f, to denote the leader and follower spacecraft, respectively.

3.1.2 Model with reference frame attached to leader spacecraft

Another possibility is to derive the equation of relative motion with respect to an origin attached to the leader spacecraft. When the reference frame is attached to the leader spacecraft, the dynamics describing the motion of the follower spacecraft, relative to the leader can be written :

$$m_f \ddot{\rho} + C_f(\dot{\nu}_l) \dot{\rho} + D_f(\dot{\nu}_l, \ddot{\nu}_l, r_l, \rho) \rho + n_f(r_l, \rho) = u_f + d_f - \frac{m_f}{m_l}(u_l + d_l), \quad (3.9)$$

with (borrowing the notation from Kristiansen et al. (2006b))

$$C_f(\dot{\nu}_l) := 2 m_f \dot{\nu}_l \bar{C}, \quad \bar{C} := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{SS}(3),$$

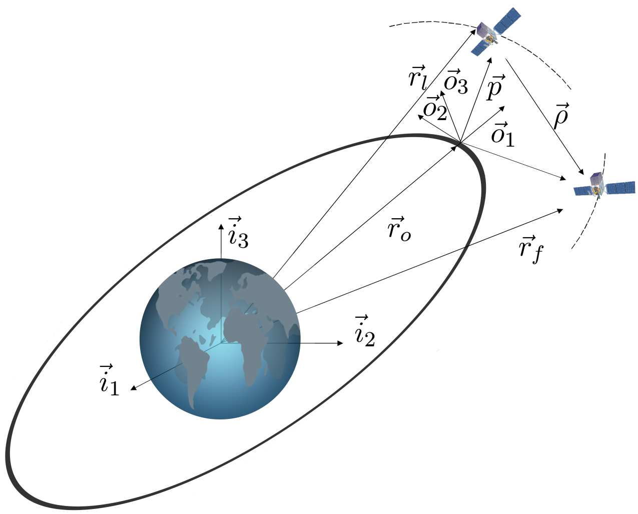

Figure 3.1: The model described in Section 3.1.1

$$D_f\left(\dot{\nu}_l, \ddot{\nu}_l, r_l, \rho\right) := m_f \dot{\nu}_l^2 \bar{D} + m_f \ddot{\nu}_l \bar{C} + m_f \frac{\mu}{\left|r_l + \rho\right|^3} I_{3\times 3},$$

$\bar{D} := \mathrm{diag}(-1, -1, 0) \in \mathbb{R}^{3\times 3}$, and

$$n_f\left(r_l, \rho\right) := m_f \mu \left(\frac{r_l}{\left|r_l + \rho\right|^3} - \frac{r_l}{\left|r_l\right|^3} \right).$$

This model is the same as in (Kristiansen et al. (2007)), and is the most commonly used model in the literature for the relative translation of spacecraft. On the downside ν_l, and its time derivatives, are no longer known a priori, but are parameters depending on the motion of the leader spacecraft. To see why, notice that the angular velocity vector is given by

$$\omega_{il}^i = \frac{S\left(r_l^i\right) \dot{r}_l^i}{\left|r_l\right|^2},$$

From the direction cosine matrix, we see that

$$\begin{aligned} \omega_{il}^l &= R_i^l \omega_{il}^i \\ &= \mathrm{col}\left(0, 0, \dot{\nu}_l\right) \end{aligned}$$

with

$$\begin{aligned}\dot{\nu}_l &= \left(\frac{S\left(r_l^i\right)\dot{r}_l^i}{\left|S\left(r_l\right)\dot{r}_l\right|}\right)^\top\left(\frac{S\left(r_l^i\right)\dot{r}_l^i}{\left|r_l\right|^2}\right) \\ &= \frac{\left|S\left(r_l\right)\dot{r}_l\right|}{\left|r_l\right|^2}\end{aligned}$$

which in fact is the true anomaly rate, Sellers (2000). We see that $\dot{\nu}_l$, depends on r_l and $\dot{r}_l$, which in turn depends on the disturbance- and control forces. The true anomaly rate could of course be given in terms of orbital parameters, but it is important to have in mind, at least from a theoretical viewpoint, that for instance the eccentricity is no longer constant if the spacecraft is perturbed from its elliptic orbit. This problem is related to the fact that the eccentricity vector (Laplace-Runge-Lenz vector) is no longer conserved, since the spacecraft, and thus the reference frame, is influenced by forces that do not obey an inverse-square law.

3.1.3 Model with reference frame attached to leader spacecraft, Alternative 2

In Chapter 4, we will also consider the following model, where the relative dynamics is described in the leader spacecraft frame:

$$m_f\ddot{\rho} + C_f(\omega_{il}^l)\dot{\rho} + D_f(\omega_{il}^l, \dot{\omega}_{il}^l)\rho + n_f\left(r_l, \rho\right) = u_f + d_f - \frac{m_f}{m_l}\left(u_l + d_l\right),$$

where

$$\begin{aligned}C_f(\omega_{il}^l) &:= 2m_f(S\omega_{il}^l) \\ D_f(\omega_{il}^l, \dot{\omega}_{il}^l) &:= m_f S(\omega_{il}^l)^2 + m_f S(\dot{\omega}_{il}^l)\end{aligned}$$

and

$$n_f\left(r_l, \rho\right) := m_f\mu\left(\frac{r_l + \rho}{\left|r_l + \rho\right|^3} - \frac{r_l}{\left|r_l\right|^3}\right).$$

This model is just a rewrite of the model (3.9). ω_{il}^l and $\dot{\omega}_{il}^l$ denote the angular velocity and acceleration of the leader spacecraft reference frame, relative to the inertial frame and is given by

$$\omega_{il}^l\left(\dot{r}_l, r_l\right) = R_i^l\frac{S\left(r_l^i\right)\dot{r}_l^i}{\left|r_l\right|^2} \tag{3.10}$$

and

$$\dot{\omega}_{il}^l = R_i^l\frac{\left\{S\left(r_l^i\right)\ddot{r}_l^i\right\}\left(r_l^i\right)^\top r_l^i - 2\left\{S\left(r_l^i\right)\dot{r}_l^i\right\}\left(\dot{r}_l^i\right)^\top r_l^i}{\left|r_l\right|^4}. \tag{3.11}$$

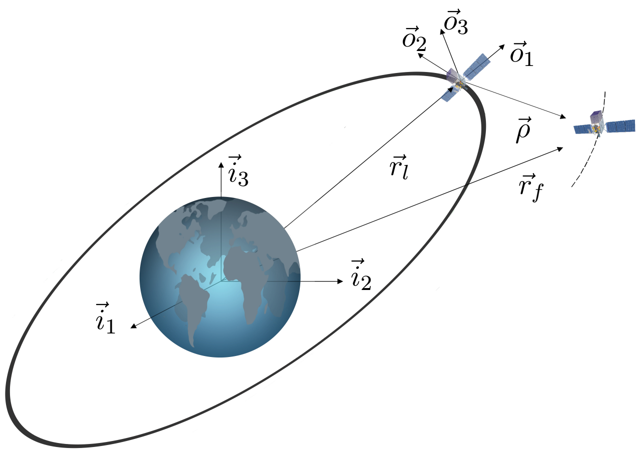

Figure 3.2: The model described in Section 3.1.3

For the leader spacecraft, the following model will be used:

$$m_l \ddot{r}_l^i + m_l \frac{\mu}{|r_l|^3} r_l^i = u_l^i + d_l^i$$

where the superscript i is used to denote that the vectors are decomposed in the inertial frame.

3.2 Rotational case

3.2.1 Model of leader spacecraft

The model for the leader spacecraft is (Hughes (1986)):

$$\dot{q}_{il} = \frac{1}{2} \begin{bmatrix} -\epsilon_{il}^\top \\ E(q_{il}) \end{bmatrix} \omega_{il}^l \tag{3.12}$$

$$J_l \dot{\omega}_{il}^l + C_l(\omega_{il}^l)\omega_{il}^l = \tau_l + d_l \tag{3.13}$$

with $J_l \in \mathbb{R}^{3\times3}$ being the leader spacecraft inertia matrix, ω_{il}^l the angular velocity of the spacecraft relative to the inertial frame, $C_l(\omega_{il}^l) = -S(J_l \omega_{il}^l)$ and τ_l and d_l the input and disturbance moments on the leader spacecraft, respectively.

3.2.2 Model of follower spacecraft

The model used for the follower spacecraft is similar to the one found in Kristiansen et al. (2007), where the model of the relative attitude in a leader-follower formation can be written as

$$\dot{q}_{lf} = \frac{1}{2} \begin{bmatrix} -\epsilon_{lf}^{\top} \\ E\left(q_{lf}\right) \end{bmatrix} \omega_{lf}^{f} \tag{3.14}$$

$$J_f \dot{\omega}_{lf}^f + C_f(\omega_{lf}^f)\omega_{lf}^f + n_f(\omega_{il}^l, \omega_{lf}^f) = \Gamma_a + \Gamma_d \tag{3.15}$$

with $J_f \in \mathbb{R}^{3\times 3}$ being the follower spacecraft inertia matrix, ω_{lf}^f the angular velocity of the follower spacecraft relative to the leader spacecraft, $C_f(\omega_{lf}^f) = -S(J_f \omega_{lf}^f)$ and

$$\begin{aligned} n_f =& (S(R_l^f \omega_{il}^l) J_f R_l^f - J_f R_l^f J_l^{-1} S(\omega_{il}^l) J_l) \omega_{il}^l \\ &+ (-S(J_f R_l^f \omega_{il}^l) + J_f S(R_l^f \omega_{il}^l) + S(R_l^f \omega_{il}^l) J_f) \omega_{lf}^f \end{aligned}$$

Furthermore,

$$\Gamma_a = \tau_f^f - J_f R_l^f J_l^{-1} \tau_l^l \tag{3.16}$$

and

$$\Gamma_d = d_f^f - J_f R_l^f J_l^{-1} d_l^l \tag{3.17}$$

with τ_f and d_f as the input and disturbance moments on the follower spacecraft, respectively.

Remark 3.3 *The model in Kristiansen et al. (2007) is slightly different, and may prove advantageous from a control design perspective, as it more thoroughly explores the properties of the matrices in the model.*

Remark 3.4 *Note that the matrices C_i, $i \in \{l, f\}$ satisfy the inequalities*

$$\left|C_i\left(a\right)b\right| \leq \left|J_i\right| \left|a\right| \left|b\right|$$

and are linear in their arguments, i.e.

$$C_i\left(\phi_1 a + \phi_2 b\right) = \phi C_i\left(a\right) + \phi C_i\left(b\right)$$

for any vectors $a, b \in \mathbb{R}^3$ and any constants $\phi_1, \phi_2 \in \mathbb{R}$.

3.3 Disturbances acting on spacecraft

In the following, models of the most prominent disturbances acting on spacecraft in orbit are listed. In addition there are several other types of disturbances that are not easily modeled, see Section 6.1.

3.3.1 Gravitational forces

The second zonal harmonic J_2 is the most dominant, and by only considering this zonal harmonic the gravitational potential can be further approximated as

$$U(r,\phi) = -\mu\left(\frac{1}{|\vec{r}|} - \frac{1}{2}J_2\frac{R_e^2}{|\vec{r}|^3}\left(3\sin^2\phi - 1\right)\right)$$

where R_e is the mean equatorial radius of the Earth, ϕ is the geocentric latitude of the spacecraft position and $\vec{r}$ is the position vector from the center of Earth to the spacecraft. Let $\vec{r}_z$ be the part of $\vec{r}$ along the line connecting the center of Earth with the geometrical north pole. Then, using that $\sin\phi = r_z/|\vec{r}|$, we get that the gravitational forces acting on a spacecraft is given by

$$\begin{aligned}\vec{f} &= -m\nabla U \\ &= -m\mu\left(-\frac{\vec{r}}{|\vec{r}|^2} - \frac{1}{2}J_2R_e^2\left(-15r_z^2\frac{\vec{r}}{|\vec{r}|^7} + 3\frac{\vec{r}_z}{|\vec{r}|^5|\vec{r}_z|} + 3\frac{\vec{r}}{|\vec{r}|^5}\right)\right).\end{aligned}$$

3.3.2 Aerodynamic drag

The forces acting on a spacecraft due to atmospheric drag, can, based on empirical observations, be modeled as Ploen et al. (2004b):

$$\vec{f}_{\text{drag}} = -\frac{1}{2}\rho C_{\text{drag}}A_{\text{eff}}\left|\vec{v}_{as}\right|\vec{v}_{as},$$

where ρ is the atmospheric density, C_{drag} is the drag coefficient, A_{eff} is the effective incident area and $\vec{v}_{as}$ is the velocity vector of the spacecraft relative to the atmosphere of the Earth. If, in addition, it is assumed that the atmosphere rotates with the Earth, then

$$\vec{v}_{as} = \frac{^i\mathrm{d}}{\mathrm{d}t}\vec{r} - \vec{\omega}_{ie}\times\vec{r}$$

with $\vec{\omega}_{ie}$ being the angular velocity of the Earth.

3.3.3 Solar radiation

The force acting on a spacecraft due to the mean solar energy flux is proportional to the inverse square of the distance from the Sun, and can according to Montenbruck and Gill (2000) be modeled as:

$$\vec{f}_{\mathrm{rad}} = -P\cos\beta A\,(1-\varepsilon)\,\vec{e} + 2\epsilon\cos\beta\vec{n},$$

with P being the solar radiation pressure, $\vec{n}$ is the normal vector of the radiated surface area A and $\vec{e}$ points in the direction of the Sun, inclined at an angle β relative to $\vec{n}$. The distance between the Sun and the spacecraft is assumed constant.

3.3.4 Third body gravitational perturbations

The gravitational perturbing forces due to $j = 1, .., N$ interacting bodies modeled as point masses, can be given as:

$$\vec{f} = \sum_{j=1}^{N} m\mu_j \left(\frac{\vec{r}_j}{|\vec{r}_j|^3} - \frac{\vec{r}_{ej}}{|\vec{r}_{ej}|} \right),$$

where μ_j is the gravitational parameter of the j^{th} perturbing body, $\vec{r}_{ej}$ is the vector from the Earth to the perturbing body j, and finally $\vec{r}_j$ is the vector from the spacecraft to the perturbing body j, such that $\vec{r}_j = \vec{r}_{ej} - \vec{r}$.

Chapter 4

Output tracking control of leader-follower formation: translational case

4.1 Control of relative motion in leader fixed coordinate frame

In this Section, as in the original publications Grøtli and Gravdahl (2007) and Grøtli et al. (2008), we only consider the relative dynamics of two spacecraft, and with the origin of the frame of reference situated at the origin of the leader spacecraft.

4.1.1 Model and desired trajectory assumptions

The relative dynamics were given in Section (3.1.2) by:

$$m_f\ddot{\rho} + C_f\left(\dot{\nu}_l\right)\dot{\rho} + D_f\left(\dot{\nu}_l, \ddot{\nu}_l, r_l, \rho\right)\rho + n_f\left(r_l, \rho\right) = u_f + d_f - \frac{m_f}{m_l}\left(u_l + d_l\right), \tag{4.1}$$

where

$$D_f\left(\dot{\nu}_l, \ddot{\nu}_l, r_l, \rho\right) := m_f\dot{\nu}_l^2\bar{D} + m_f\ddot{\nu}_l\bar{C} + m_f\frac{\mu}{\left|r_l + \rho\right|^3}I_{3\times 3}$$

and

$$n_f\left(r_l, \rho\right) := m_f\mu\left(\frac{r_l}{\left|r_l + \rho\right|^3} - \frac{r_l}{\left|r_l\right|^3}\right).$$

The true anomaly of the leader spacecraft is the angle between the eccentricity vector

$$e_l = \frac{\dot{r}_l \times h}{\mu} - \frac{r_l}{|r_l|} \in \mathbb{R}^3 \tag{4.2}$$

where $h = r_l \times \dot{r}_l \in \mathbb{R}^3$, and r_l is the orbital state vector, so that:

$$\nu_l = \begin{cases} \arccos \frac{e_l^\top r_l}{|e_l||r_l|} & \text{if} \quad r_l^\top \dot{r}_l \geq 0 \\ 2\pi - \arccos \frac{e_l^\top r_l}{|e_l||r_l|} & \text{if} \quad r_l^\top \dot{r}_l < 0 \end{cases} \tag{4.3}$$

The eccentricity vector is conserved under forces that obey the inverse-square law as e.g. the gravitational forces, but due to the control and disturbance forces in (4.1), the eccentricity vector will vary. See Egorov (1958) for a discussion of the definition of the true anomaly in perturbed motion. We choose the reference trajectory of the leader spacecraft to satisfy the inverse square law such that the eccentricity is constant. Then, the desired true anomaly rate and true anomaly rate of change of the leader spacecraft, denoted $\dot{\nu}_d$ and $\ddot{\nu}_d$, are given by:

$$\dot{\nu}_d(t) = \frac{n_d(1 + e_d \cos \nu_d(t))^2}{(1 - e_d^2)^{\frac{3}{2}}} \tag{4.4}$$

and

$$\ddot{\nu}_d(t) = \frac{-2n_d^2 e_d(1 + e_d \cos \nu_d(t))^3 \sin \nu_d(t)}{(1 - e_d^2)^3}, \tag{4.5}$$

with $n_d = \sqrt{\mu / a_d^3} \in \mathbb{R}$ as the desired mean motion of the leader, and $a_d \in \mathbb{R}$ and $e_d \in \mathbb{R}$ as the semimajor axis and the eccentricity of the desired spacecraft orbit, respectively. The desired trajectory is an elliptic orbit around the Earth, and hence $e_d \in (0, 1)$.

We will make some basic assumptions with respect to the motion of the leader spacecraft and the reference trajectory of the follower spacecraft. Stability analysis will be based on different assumptions on the degree of information about the leader spacecraft, which is summarized in Assumption 4.1 and 4.2.

Assumption 4.1 *The true anomaly rate, $\dot{\nu}_l$, and the true anomaly rate-of-change, $\ddot{\nu}_l$, of the leader spacecraft are assumed to be known, and, given a positive constant r, $|\dot{\nu}_l(0)| \leq r$ implies that $|\dot{\nu}_l(t)| \leq \beta_{\dot{\nu}_l}$, $t \geq t_0 \geq 0$.*

The bounds on $\dot{\nu}_l(t)$ is a fair assumption for a leader spacecraft in orbit. Notice that $|\dot{\nu}_l(0)| \leq r$ restrict the initial conditions of the leader spacecraft. It is, however, not a restriction on the initial relative position/velocity/acceleration of the two spacecraft. We now state an assumption which is a relaxation of the previous. We will no longer assume exact knowledge of $\dot{\nu}_l$ and $\ddot{\nu}_l$. Instead we assume that the control of the leader spacecraft is sufficiently good, such that even under disturbances the following hold:

Assumption 4.2 *Define $\tilde{\nu}(t) := \nu_l - \nu_d$, where $\nu_l(t)$ and $\nu_d(t)$ are the actual and the desired true anomaly of the leader spacecraft, respectively. We will assume that the desired true anomaly rate of the leader spacecraft is bounded, i.e. given a positive constant r_1, $|\dot{\nu}_d(t_0)| \leq r_1$ implies that $|\dot{\nu}_d(t)| \leq \beta_{\dot{\nu}_d}$ for all $t \geq t_0 \geq 0$, for some positive constant $\beta_{\dot{\nu}_d}$. Furthermore, we assume that the actuation system of the leader spacecraft keeps $\dot{\tilde{\nu}}$ and $\ddot{\tilde{\nu}}$ bounded, i.e. given some positive constant r_2, r_3, $\left|\dot{\tilde{\nu}}(t_0)\right| \leq r_2$ implies that $\left|\dot{\tilde{\nu}}(t)\right| \leq \beta_{\dot{\tilde{\nu}}}$ for all $t \geq t_0 \geq 0$, and $\left|\ddot{\tilde{\nu}}(t_0)\right| \leq r_3$ implies that $\left|\ddot{\tilde{\nu}}(t)\right| \leq \beta_{\ddot{\tilde{\nu}}}$ for all $t \geq t_0 \geq 0$, where $\beta_{\dot{\tilde{\nu}}}, \beta_{\ddot{\tilde{\nu}}}$ are positive constants.*

Again we emphasize that this is no restriction on the initial relative state vector, to be defined in the sequel.

In addition we will make the following assumptions regarding the desired trajectories of the follower spacecraft:

Assumption 4.3 *The desired relative position $\rho_d(t)$, desired relative velocity $\dot{\rho}_d(t)$ and desired relative acceleration $\ddot{\rho}_d(t)$ are all smooth and bounded functions, i.e. there exists positive constants β_{ρ_d}, $\beta_{\dot{\rho}_d}$, $\beta_{\ddot{\rho}_d}$ such that $|\rho_d(t)| \leq \beta_{\rho_d}$, $|\dot{\rho}_d(t)| \leq \beta_{\dot{\rho}_d}$ and $|\ddot{\rho}_d(t)| \leq \beta_{\ddot{\rho}_d}$ for all $t \geq t_0 \geq 0$.*

Finally, we assume that the disturbances acting on the spacecraft are bounded.

Assumption 4.4 *The disturbances acting on the follower spacecraft are bounded, i.e. there exist a positive constant β_{d_f} such that*

$$|d_f(t)| \leq \beta_{d_f} \tag{4.6}$$

and that the difference between thrust and external disturbances acting on the leader spacecraft is bounded, that is:

$$|u_l(t) + d_l(t)| \leq \beta_{(u_l+d_l)} \tag{4.7}$$

for a positive constant $\beta_{(u_l+d_l)}$.

4.1.2 Controller scheme

In this section the controller scheme of Paden and Panja (1988) as redefined for output feedback in Berghuis and Nijmeijer (1993) will be used.

Without disturbance

Define $e_f := \rho - \rho_d \in \mathbb{R}^3$ as the position error and $\tilde{\rho} := \rho - \hat{\rho}$ as the observer estimation error. Let the controller of the follower spacecraft be:

$$\begin{aligned} u_f &= m_f \ddot{\rho}_d + C_f(\dot{\nu}_l)\dot{\rho}_d + D_f(\dot{\nu}_l, \ddot{\nu}_l, r_l, \rho)\,\rho \\ &\quad + n_f(r_l, \rho) - K_d(\dot{\rho}_0 - \dot{\rho}_r) & (4.8) \\ \dot{\rho}_r &= \dot{\rho}_d - \Lambda_f e_f & (4.9) \\ \dot{\rho}_0 &= \dot{\hat{\rho}} - \Lambda_f \tilde{\rho}, & (4.10) \end{aligned}$$

where $\Lambda_f = \Lambda_f^\top \in \mathbb{R}^{3\times 3} > 0$, $K_d := k_d I_{3\times 3}$ with $k_d > m_f \lambda_{\max}(\Lambda_f) + 2m_f \beta_{\dot{\nu}_l}$. Let the observer be:

$$\begin{aligned} \dot{\hat{\rho}} &= a_f + (l_f I + \Lambda_f)\,\tilde{\rho} & (4.11) \\ \dot{a}_f &= \ddot{\rho}_d + l_f \Lambda_f \tilde{\rho}, & (4.12) \end{aligned}$$

with $l_f > 2k_d/m_f$ being a scalar.

The following Proposition was given in (Grøtli and Gravdahl, 2007, Proposition 2).

Proposition 4.1 *Let $|\dot{\nu}_l(0)| \leq r$ for some positive constant r. Let Assumption 4.1 and 4.3 hold. Assume that $u_l + d_l = 0$, $d_f = 0$. Then the origin of (4.1), in closed loop with the controller (4.8-4.10) and the observer (4.11-4.12) is uniformly globally exponentially stable.*

Proof. By inserting (4.8-4.10) into (4.1), the closed-loop tracking error dynamics are found to be

$$m_f \ddot{e}_f + C_f(\dot{\nu}_l)\dot{e}_f + K_f(\dot{\rho}_0 - \dot{\rho}_r) = 0, \qquad (4.13)$$

since $\rho - \rho_d = e_f$. Now, defining the sliding variables $t_1, t_2 \in \mathbb{R}^3$ as

$$\begin{aligned} t_1 &:= \dot{\rho} - \dot{\rho}_r = \dot{e}_f + \Lambda_f e_f & (4.14) \\ t_2 &:= \dot{\rho} - \dot{\rho}_0 = \dot{\tilde{\rho}} + \Lambda_f \tilde{\rho}, & (4.15) \end{aligned}$$

we get the tracking error dynamics

$$m_f \dot{t}_1 = m_f \Lambda_f \dot{e}_f - C_f(\dot{\nu}_l)\dot{e}_f - K_f(t_1 - t_2), \qquad (4.16)$$

since $\dot{\rho}_0 - \dot{\rho}_r = t_1 - t_2$. The observer error dynamics is

$$m_f \dot{t}_2 = -C_f(\dot{\nu}_l)\dot{e}_f - K_f(t_1 - t_2) - m_f l_f t_2. \tag{4.17}$$

Let the Lyapunov function candidate be given by (cf. Berghuis and Nijmeijer (1994) and Berghuis and Nijmeijer (1993))

$$V(x) := \frac{1}{2} x^\top W^\top R W x, \tag{4.18}$$

where $x := (\dot{e}_f^\top, (\Lambda_f e_f)^\top, \dot{\tilde{\rho}}^\top, (\Lambda_f \tilde{\rho})^\top) \in \mathbb{R}^{12}$, $R := \mathrm{diag}(m_f I_{3\times3}, 2K_f\Lambda_f^{-1} - m_f I_{3\times3}, m_f I_{3\times3}, 2K_f\Lambda_f^{-1}) \in \mathbb{R}^{12\times12}$ and

$$W := \begin{bmatrix} I_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_{3\times3} & I_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3} \end{bmatrix} \in \mathbb{R}^{12\times12}. \tag{4.19}$$

Note that for $K_f > m_f \Lambda_f$, we have that

$$k_1 |x|^2 \le V \le k_2 |x|^2$$

with $k_1 = \frac{1}{6}\lambda_{\min}(R)$ and $k_2 = \frac{3}{2}\lambda_{\max}(R)$, where $\lambda_{\min}(R) = m_f$ and $\lambda_{\max}(R) = 2k_f\lambda_{\min}(\Lambda_f)^{-1}$. This can be verified using the fact that $\frac{1}{3} \le \lambda_{\min}(W^\top W)$ and that $\lambda_{\max}(W^\top W) \le 3$, where $\lambda_{\min}(W^\top W)$ and $\lambda_{\max}(W^\top W)$ denote the minimum and maximum eigenvalue of $W^\top W$, respectively. The time derivative of the Lyapunov function candidate along the error dynamics (4.16) and (4.17) is

$$\begin{aligned} \dot{V} = &- x^\top Q x - t_2^\top (l_f m_f I_{3\times3} - 2K_f) t_2 \\ &- (t_1 + t_2)^\top C_f(\dot{\nu}_l)\dot{e}_f, \end{aligned} \tag{4.20}$$

where $Q := \mathrm{diag}(K_f - m_f\Lambda_f, K_f, K_f, K_f) \in \mathbb{R}^{12\times12}$. By using that $l_f \ge 2k_f/m_f$ and that $|\dot{\nu}_l(t)| \le \beta_{\dot{\nu}_l}$ and $\left|\dot{\tilde{\nu}}(t)\right| \le \beta_{\dot{\tilde{\nu}}}$, we get that

$$\begin{aligned} \dot{V} &\le -(k_f - m_f\lambda_{\max}(\Lambda_f) - 2m_f\beta_{\dot{\nu}_l}) |x|^2 \\ &\le -k_3 |x|^2, \end{aligned} \tag{4.21}$$

where k_3 is a positive constant. It has also been used that $\left|(t_1 + t_2)^\top C_f(\dot{\nu}_l)\dot{e}_f\right| \le |t_1 + t_2| \, |C_f(\dot{\nu}_l)\dot{e}_f| \le 2m_f\beta_{\dot{\nu}_l} |x|^2$ where $(t_1 + t_2) = Yx$ with

$$Y := \begin{bmatrix} I_{3\times3} & I_{3\times3} & I_{3\times3} & I_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times12},$$

so that $|Yx| \le |Y| \, |x| = 2|x|$. Hence, according to (Khalil, 2002, Theorem 4.10), the origin of the system is UGES. ■

With disturbances

In the previous section it is assumed that the true anomaly, $\dot{\nu}_l$, and true anomaly rate of change, $\ddot{\nu}_l$, of the leader spacecraft are available. Since these parameters can be considered as velocity and acceleration parameters, we will now treat the case where the true values of $\dot{\nu}_l$ and $\ddot{\nu}_l$ are unknown.

Let the controller of the follower spacecraft be:

$$\begin{aligned} u_f =& m_f\ddot{\rho}_d + C_f(\dot{\nu}_d)\dot{\rho}_d + D_f\left(\dot{\nu}_d, \ddot{\nu}_d, r_l, \rho\right)\rho \\ &+ n_f\left(r_l, \rho\right) - K_f(\dot{\rho}_0 - \dot{\rho}_r) \end{aligned} \tag{4.22}$$

$$\dot{\rho}_r = \dot{\rho}_d - \Lambda_f e_f \tag{4.23}$$

$$\dot{\rho}_0 = \dot{\hat{\rho}} - \Lambda_f\tilde{\rho}, \tag{4.24}$$

with observer (4.11-4.12). Note the difference in equation (4.22) from that of equation (4.8) in that the parameters $\dot{\nu}_d, \ddot{\nu}_d$, of the desired trajectory of the leader spacecraft are used, instead of the actual parameters, $\dot{\nu}_l, \ddot{\nu}_l$, of the leader spacecraft orbit. By using the error of the true anomaly, $\tilde{\nu} = \nu_l - \nu_d$, we get that the tracking error dynamics are

$$\begin{aligned} m_f\ddot{e}_f =& - C_f(\dot{\nu}_l)\dot{e}_f - K_f(t_1 - t_2) \\ &+ d_f - \frac{m_f}{m_l}\left(u_l + d_l\right) - 2m_f\bar{C}\dot{\tilde{\nu}}\dot{\rho}_d \\ &- m_f\bar{D}\left(\dot{\tilde{\nu}}^2 + 2\dot{\tilde{\nu}}\dot{\nu}_d\right)\rho_d - m_f\bar{C}\ddot{\tilde{\nu}}\rho_d. \end{aligned} \tag{4.25}$$

Similarly the observer error dynamics, using the observer (4.11) and (4.12) become

$$\begin{aligned} m_f\ddot{\tilde{\rho}} =& - m_f\Lambda_f\dot{\tilde{\rho}} - C_f(\dot{\nu}_l)\dot{e}_f - K_f(t_1 - t_2) \\ &- m_f l_f t_2 + d_f - \frac{m_f}{m_l}(u_l + d_l) - 2m_f\bar{C}\dot{\tilde{\nu}}\dot{\rho}_d \\ &- m_f\bar{D}\left(\dot{\tilde{\nu}}^2 + 2\dot{\tilde{\nu}}\dot{\nu}_d\right)\rho_d - m_f\bar{C}\ddot{\tilde{\nu}}\rho_d. \end{aligned} \tag{4.26}$$

Proposition 4.2 *Let* $|\dot{\nu}_d\,(t_0)| \leq r_1$, $\left|\dot{\tilde{\nu}}\,(t_0)\right| \leq r_2$ *and* $\left|\ddot{\tilde{\nu}}\,(t_0)\right| \leq r_3$ *for some positive constants* r_1, r_2 *and* r_3. *Let Assumption 4.2-4.4 hold. The controller given by (4.22)-(4.24) and observer (4.11)-(4.12) in closed loop with (4.1) is UGPES, with* $\theta = (k_f, l_f)^\top \in \Theta$ *as tuning parameters.*

Proof. *The proof is done by applying Theorem 2.1. Using (4.18) as the Lyapunov function candidate, we get that its time derivative along (4.25)* *and (4.26) is bounded as:*

$$\begin{aligned}\dot{V} \leq &- \left(k_f - m_f\lambda_{\max}(\Lambda_f) - 2m_f(\beta_{\dot{\tilde{\nu}}} + \beta_{\dot{\nu}_d})\right)|x|^2 \\ &+ 2\Big(\beta_{d_f} + \frac{m_f}{m_l}\beta_{(u_l+d_l)} + 2m_f\beta_{\dot{\tilde{\nu}}}\beta_{\dot{\rho}_d} \\ &+ m_f(\beta_{\dot{\tilde{\nu}}}^2 + 2\beta_{\dot{\tilde{\nu}}}\beta_{\dot{\nu}_d} + \beta_{\ddot{\tilde{\nu}}})\beta_{\rho_d}\Big)|x|,\end{aligned} \tag{4.27}$$

by similar calculations as in the proof of Proposition 4.1. Let δ be any positive constant. Pick $l_f \geq l_f^\star := 2k_f/m_f$. Pick $k_f \geq k_f^\star$, where

$$\begin{aligned}k_f^\star :=& 2m_f\lambda_{\max}(\Lambda_f) + 4m_f(\beta_{\dot{\tilde{\nu}}} + \beta_{\dot{\nu}_d}) \\ &+ \frac{4}{\delta}\Big(\beta_{d_f} + \frac{m_f}{m_l}\beta_{(u_l+d_l)} + 2m_f\beta_{\dot{\tilde{\nu}}}\beta_{\dot{\rho}_d} \\ &+ m_f(\beta_{\dot{\tilde{\nu}}}^2 + 2\beta_{\dot{\tilde{\nu}}}\beta_{\dot{\nu}_d} + \beta_{\ddot{\tilde{\nu}}})\beta_{\rho_d}\Big)\end{aligned} \tag{4.28}$$

Then, for any $|x| \geq \delta$ we have that

$$\dot{V} \leq -\frac{1}{2}k_f^\star|x|^2 \tag{4.29}$$

and we can apply Theorem 2.1 with $p = 2$, $V_\delta = V$, $\underline{\kappa}(\delta) = \frac{1}{6}\lambda_{\min}(R) = \frac{1}{6}m_f$, $\overline{\kappa} = \frac{3}{2}\lambda_{\max}(R(\delta)) = 3k_f^\star(\delta)/\lambda_{\max}(\Lambda_f)$ and $\kappa(\delta) = \frac{1}{2}k_f^\star(\delta)$. Finally we have

$$\lim_{\delta\to 0}\frac{\overline{\kappa}(\delta)\delta^p}{\underline{\kappa}(\delta)} = \lim_{\delta\to 0}\frac{18k_f^\star(\delta)\delta^2}{\lambda_{\max}(\Lambda_f)m_f} = 0, \tag{4.30}$$

thus (2.7) is also satisfied and we can conclude UGPES of model (4.1), in closed loop with the controller (4.8), (4.9), (4.10) and observer (4.11), (4.12). ■

Remark 4.1 *Although we focused on robustness with respect to external disturbances and inaccurate control of the leader spacecraft in this section, the analysis can easily be extended to also include robustness with respect to uncertainties in the spacecraft masses.*

4.1.3 Simulations

In this section the performance of the controller-observer scheme will be illustrated by simulations. The desired orbit of the leader spacecraft is of eccentricity $e_d = 0.5$, and with semimajor axis $a_d = 20000$ km. The true anomaly rate and true anomaly rate-of-change are generated by (4.4)

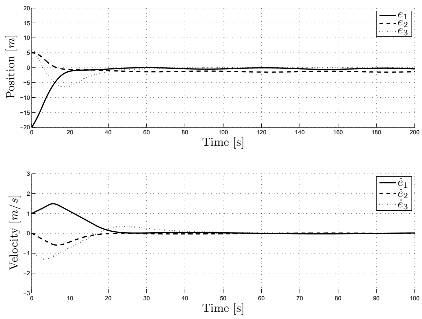

Figure 4.1: Position and velocity tracking errors

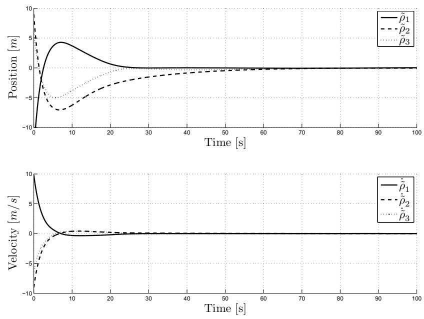

Figure 4.2: Position and velocity estimation errors

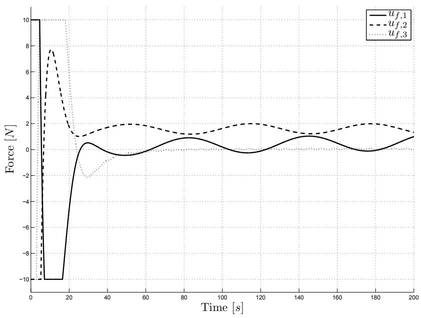

Figure 4.3: Control history

and (4.5). We want to illustrate the robustness of our controller-observer scheme even under perturbed motion of the leader spacecraft. For that reason the leader spacecraft is simulated according to

$$\ddot{r}_l = -\frac{\mu}{|r_l|^3} r_l + \frac{d_l + u_l}{m_l}$$

with $u_l + d_l = \text{col}(0.5 \sin \frac{1}{10}t, 0.2 \sin \frac{1}{100}t, 0.3 \sin \frac{1}{1000}t)$ to illustrate a control system that is not able to handle the periodic forces that an orbiting spacecraft are exposed to. The true anomaly rate and rate-of-change of the leader spacecraft are achieved by differentiation of (4.3). The desired trajectory of the follower spacecraft is given by $\rho_d(t) = \text{col}(-10 \cos \nu, 20 \sin \nu, 0)$, which means that the follower spacecraft evolves around the leader spacecraft in an ellipse during their orbit around the Earth. This is a fuel efficient orbit, as it is close to a natural orbit of the spacecraft. We assume that the follower spacecraft is exposed to similar perturbations as the leader spacecraft, and we have chosen that $d_f = \mathrm{col}(0.1 \sin \frac{1}{100}t, 0.3 \sin \frac{1}{10}t, 0.4 \sin t)$. The initial position and velocity of the follower spacecraft is chosen as $\rho(0) = \mathrm{col}(-10, 5, 7)$ and $\dot{\rho}(0) = \mathrm{col}(1, 0, -1)$, where as the initial states of the observer are $\hat{\rho}(0) = \mathrm{col}(4, -4, 1)$ and $a_f(0) = \mathrm{col}(-1, 4, 2)$. The controller and observer gains are as follows: $l_f = 0.5$, $K_d = 20I_{3\times3}$, $\Lambda_f = 0.06I_{3\times3}$. Both spacecraft are of mass $m_l = m_f = 100\,\mathrm{kg}$. Furthermore, the thrust is assumed to be continuous and available in all directions of the leader spacecraft frame, but limited to $\max u_f = 10\,\mathrm{N}$. Figure 4.1 and 4.2 show the tracking and estimation errors, respectively. As proven in the previous section, the tracking error in Figure 4.1 can be arbitrarily diminished by an appropriate choice of control gains, e.g. by increasing K_d. The control history is shown in Figure 4.3. The actuation of the follower spacecraft would be greatly reduced by a better controlled leader spacecraft, as we use $\dot{\nu}_d$ and $\ddot{\nu}_d$, instead of the actual parameters $\dot{\nu}_l$ and $\ddot{\nu}_l$ for the true anomaly rate and rate-of-change. To further save fuel, one can imagine that control parameters are changed so as extensive actuation is used only when high accuracy formation control is needed, e.g. only during performance of measurement.

4.2 Control of a leader-follower spacecraft formation in leader fixed coordinate frame

4.2.1 Model of the formation

In this section we will use the model of Section 3.1.3. For the leader spacecraft we use that

$$m_l \ddot{r}_l^i + m_l \frac{\mu}{|r_l|^3} r_l^i = u_l^i + d_l^i$$

where r_l is the position of the leader spacecraft with respect to the center of Earth, m_l is the mass of the spacecraft, and u_l and d_l are the control- and disturbance forces acting on the spacecraft. The relative dynamics describing the motion of the follower spacecraft is given by

$$m_f \ddot{\rho} + C_f(\omega_{il}^l)\dot{\rho} + D_f(\omega_{il}^l, \dot{\omega}_{il}^l)\rho + n_f\left(r_l, \rho\right) = u_f + d_f - \frac{m_f}{m_l}\left(u_l + d_l\right),$$

where

$$C_f(\omega_{il}^l) := 2m_f S(\omega_{il}^l) \tag{4.31}$$

$$D_f(\omega_{il}^l, \dot{\omega}_{il}^l) := m_f S(\omega_{il}^l)^2 + m_f S(\dot{\omega}_{il}^l)$$

and

$$n_f\left(r_l, \rho\right) := m_f \mu \left(\frac{r_l + \rho}{\left|r_l + \rho\right|^3} - \frac{r_l}{\left|r_l\right|^3} \right).$$

m_f is the mass of the follower spacecraft, u_f and d_f are the control- and disturbance forces, respectively. ω_{il}^l and $\dot{\omega}_{il}^l$ denotes the angular velocity and acceleration of the leader spacecraft reference frame, relative to the inertial frame and is given by

$$\omega_{il}^l\left(\dot{r}_l, r_l\right) = R_i^l \frac{S\left(r_l^i\right) \dot{r}_l^i}{\left|r_l\right|^2} \tag{4.32}$$

and

$$\dot{\omega}_{il}^l = R_i^l \frac{\left\{S\left(r_l^i\right) \ddot{r}_l^i\right\} (r_l^i)^\top r_l^i - 2\left\{S\left(r_l^i\right) \dot{r}_l^i\right\} (\dot{r}_l^i)^\top r_l^i}{\left|r_l\right|^4}. \tag{4.33}$$

The superscript i and l denoting the frame of reference in which the vector is decomposed, is left out of notational simplicity, when there is no peril of confusion.

4.2.2 Assumptions on reference trajectories

We will make use of the following assumptions:

Assumption 4.5 *Let the leader spacecraft reference trajectory be given by $r_d(t)$. We will assume that r_d is smooth and that there exist positive constants scalars $\alpha_{r_d}, \beta_{r_d}, \beta_{\dot{r}_d}, \beta_{\ddot{r}_d}$ such that*

$$\alpha_{r_d} \le \left|r_d(t)\right| \le \beta_{r_d},$$

$$\left|\dot{r}_d(t)\right| \le \beta_{\dot{r}_d},$$

$$\left|\ddot{r}_d(t)\right| \le \beta_{\ddot{r}_d},$$

for all $t \ge t_0 \ge 0$. Similarly, the reference trajectory of the follower spacecraft, $\rho_d(t)$ is assumed to be smooth and bounded such that

$$\alpha_{p_d} \le \left|\rho_d(t)\right| \le \beta_{p_d},$$

$$\left|\dot{\rho}_d(t)\right| \le \beta_{\dot{p}_d},$$

$$\left|\ddot{\rho}_d(t)\right| \le \beta_{\ddot{p}_d},$$

for some positive scalars $\alpha_{\rho_d}, \beta_{\rho_d}, \beta_{\dot{\rho}_d}, \beta_{\ddot{\rho}_d}$.

Assumption 4.6 *The disturbances acting on the formation are bounded, i.e. there exists positive scalars β_{d_l} and β_{d_f} such that*

$$|d_l(t)| \leq \beta_{d_l}$$

and

$$|d_f(t)| \leq \beta_{d_f},$$

for all $t \geq t_0 \geq 0$.

Assumption 4.7 *The vector describing the motion of the leader spacecraft relative to the center of Earth, is bounded from below by a positive constant. Out of simplicity we will use that*

$$|r_l(t)| \geq 1,$$

for all $t \geq t_0 \geq 0$.

Remark 4.2 *Assumption 4.7 deserves a few comments. The proof of convergence of the controllers in this section rely on this assumption, and since $e_l = r_l - r_d$ will be used as one of the state variables, the claim of* global *stability results may seem inappropriate. The argument for still claiming global results, is that there is a physical limitation, i.e. the Earths radius r_E, such that $|r_l(t)| \geq r_E$ for all $t \geq t_0 \geq 0$, and Assumption 4.7 holds.*

4.2.3 Control schemes

Control of leader spacecraft

Let $\hat{r}$ be the estimated position of the leader spacecraft. Define $e_l := r_l - r_d$ as the position error, and $\tilde{r} := r_l - \hat{r}$ as the estimation error. For the leader spacecraft to track its desired position $r_d(t)$ we use the controller

$$u_l = m_l \ddot{r}_d + m_l \frac{\mu}{|r_l|^3} r_l - K_l (\dot{r}_0 - \dot{r}_r) \tag{4.34}$$

$$\dot{r}_r = \dot{r}_d - \Lambda_l e_l \tag{4.35}$$

$$\dot{r}_0 = \dot{\hat{r}} - \Lambda_l \tilde{r} \tag{4.36}$$

and observer

$$\dot{\hat{r}} = a_l + (l_l I_{3\times 3} + \Lambda_l)\tilde{r} \tag{4.37}$$

$$\dot{a}_l = \ddot{r}_d + l_l \Lambda_l \tilde{r} \tag{4.38}$$

where $K_l \in \mathbb{R}^{3\times3}$ is a positive definite matrix and $\Lambda_l \in \mathbb{R}^{3\times3}$ is a symmetric, positive definite matrix. For simplicity we choose $K_l := k_l I_{3\times3}$.

Define $x_2 := (\dot{e}_l^\top, (\Lambda_l e_l)^\top, \dot{\tilde{r}}^\top, (\Lambda_l \tilde{r})^\top)^\top \in \mathbb{R}^{12}$ and $\theta_2 := (k_l, l_l)^\top \in \mathbb{R}^2$. By defining

$$Y := \begin{bmatrix} I_{3\times3} & I_{3\times3} & -I_{3\times3} & -I_{3\times3} \end{bmatrix}$$

we can write $\dot{r}_0 - \dot{r}_r = Yx_2$. We can now write the leader spacecraft closed-loop system on state space form:

$$\dot{x}_2 = f_2(t, x_2, \theta_2) \tag{4.39}$$

where

$$f_2(t, x_2, \theta_2) := \begin{bmatrix} m_l^{-1}\sigma_{21} \\ \Lambda_l \dot{e}_l \\ m_l^{-1}\sigma_{23} \\ \Lambda_l \dot{\tilde{r}} \end{bmatrix} \tag{4.40}$$

with

$$\sigma_{21} := -K_l Y x_2 + d_l$$

and

$$\sigma_{23} := -m_l \Lambda_l \dot{\tilde{r}} - K_l Y x_2 + d_l - m_l l_l \left(\dot{\tilde{r}} + \Lambda_l \tilde{r}\right)$$

We are now ready to state the following proposition

Proposition 4.3 *Under Assumption 4.5 and 4.6, the system (4.39) is UGPES with θ_2 as tuning parameter.*

Proof. The proof is done by applying Theorem 2.1. We choose the Lyapunov function candidate as

$$V_2(x_2) := x_2^\top W^\top R_2 W x_2$$

where $R_2 := \mathrm{diag}(m_l I_{3\times3}, 2K_l\Lambda_l^{-1} - m_l I_{3\times3}, m_l I_{3\times3}, 2K_l\Lambda_l^{-1}) \in \mathbb{R}^{12\times12}$ and

$$W := \begin{bmatrix} I_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_{3\times3} & I_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3} \end{bmatrix} \in \mathbb{R}^{12\times12}$$

We define the sliding variables:

$$s_1 := \dot{r}_l - \dot{r}_r$$
$$s_2 := \dot{r}_l - \dot{r}_0$$

such that $s_1 = \dot{e}_l + \Lambda_l e_l$ and $s_2 = \dot{\tilde{r}} + \Lambda_l \tilde{r}$, to aid the Lyapunov analysis. It can be shown that the time derivative of the Lyapunov function candidate can be compactly written as

$$\dot{V}_2 = -x_2^\top Q_l x_2 - s_2^\top \left(l_l m_l I_{3\times 3} - 2K_l\right) s_2 + m_l(s_1 + s_2) d_l$$

where $Q_l := \mathrm{diag}(K_l - m_l\Lambda_l, K_l, K_l, K_l) \in \mathbb{R}^{12\times 12}$. By picking $l_l \geq l_l^\star := 2k_l/m_l$, and using Assumption 4.6 and that $|s_1 + s_2| \leq 2\,|x_2|$ we get

$$\dot{V}_2 \leq -\left(k_l - m_l \lambda_{\max}\left(\Lambda_l\right)\right)|x_2|^2 + 2m_l\beta_{d_l}\,|x_2|\,.$$

Let δ_2 be any positive constant. Pick $k_l \geq k_l^\star := 2m_l\lambda_{\max}\left(\Lambda_l\right) + 4m_l\beta_{d_l}/\delta_2$. Then for any $|x_2| \geq \delta_2$, we have that

$$\dot{V}_2 \leq -\frac{1}{2}k_l^\star\,|x_2|^2$$

Notice that this choice of k_l also ensures that V_2 is positive definite. We see that (2.5-2.6) of Theorem 2.1 are satisfied with $p = 2$, $\underline{\kappa}(\delta) = 1/6\lambda_{\min}\left(R_2\right) = 1/6m_l$, $\overline{\kappa}(\delta) = 3/2\lambda_{\max}\left(R_2\left(\delta_2\right)\right) = 3k_l^\star\left(\delta_2\right)/\lambda_{\max}\left(\Lambda_l\right)$, $V_\delta = V_2$ and $\kappa(\delta) = 1/2k_l^\star\left(\delta_2\right)$. Finally, we have that

$$\lim_{\delta\to 0}\frac{\overline{\kappa}(\delta)\delta^p}{\underline{\kappa}(\delta)} = \lim_{\delta_2\to 0}\frac{18k_l^\star\left(\delta_2\right)\delta_2^2}{\lambda_{\max}\left(\Lambda_l\right)m_l} = 0,$$

thus (2.7) is also satisfied and the conclusion follows. ■

Control of follower spacecraft

To make the follower spacecraft follow the trajectory given by desired (relative) position $\rho_d\left(t\right)$, we propose a similar control algorithm as for the leader spacecraft. Let $\hat{\rho} \in \mathbb{R}^3$ be the estimated position. We define $e_f := \rho - \rho_d$ as the tracking error and $\tilde{\rho} := \rho - \hat{\rho}$ as the estimation error. In case an estimate or a measure of the angular velocity, ω_{il}^l, and angular acceleration, $\dot{\omega}_{il}^l$, of the leader spacecraft frame is available, we propose the following controller:

$$\begin{aligned} u_f &= m_f\ddot{\rho}_d + \frac{1}{2}C_f(\omega_{il}^l)(\dot{\rho}_0 + \dot{\rho}_r) + D_f(\omega_{il}^l, \dot{\omega}_{il}^l)\rho \\ &\quad + n_f\left(r_l, \rho\right) - K_f\left(\dot{\rho}_0 - \dot{\rho}_r\right) \qquad (4.41)\\ \dot{\rho}_r &= \dot{\rho}_d - \Lambda_f e_f \qquad (4.42)\\ \dot{\rho}_0 &= \dot{\hat{\rho}} - \Lambda_f\tilde{\rho}, \qquad (4.43)\end{aligned}$$

and relative position

$$\dot{\hat{\rho}} = a_f + (l_f I_{3\times 3} + \Lambda_f)\,\tilde{\rho} \tag{4.44}$$

$$\dot{a}_f = \ddot{\rho}_d + l_f \Lambda_f \tilde{\rho} \tag{4.45}$$

Similar as for the leader spacecraft we define the gain matrix $K_l \in \mathbb{R}^{3\times 3}$ as $K_l := k_l I_{3\times 3}$, a diagonal matrix with the tuning parameter k_l along the diagonal. The other tuning parameter is l_l.

Define $x_1 := (\dot{e}_f^\top, (\Lambda_f e_f)^\top, \dot{\tilde{\rho}}^\top, (\Lambda_f \tilde{\rho}_f)^\top)^\top \in \mathbb{R}^{12}$ and $\theta_1 := (k_l, l_l)^\top \in \mathbb{R}^2$. We can now write the system on a state space form $\dot{x}_1 = f_1(t, x_1, \theta_1) + g(t, x, \theta)$ where

$$f_1(t, x_1, \theta_1) := \begin{bmatrix} \frac{1}{m_f}\sigma_{11} \\ \Lambda_f \dot{e}_f \\ \frac{1}{m_f}\sigma_3 \\ \Lambda_f \dot{\tilde{\rho}} \end{bmatrix} \tag{4.46}$$

with

$$\sigma_{11} := -K_f Y x_1$$

$$\sigma_{13} := -m_f \Lambda_f \dot{\tilde{\rho}} - K_f Y x_1 - m_f l_f \left(\Lambda_f \tilde{\rho} + \dot{\tilde{\rho}}\right)$$

and where

$$g(t, x, \theta) = \begin{bmatrix} \frac{1}{m_f}\bar{g} \\ 0 \\ \frac{1}{m_f}\bar{g} \\ 0 \end{bmatrix} \tag{4.47}$$

with

$$\bar{g} := C_f(\omega_{il}^l)(\dot{\rho} - \frac{1}{2}(\dot{\rho}_0 + \dot{\rho}_r)) + d_f^l - \frac{m_f}{m_l} d_l^l - \frac{m_f}{m_l} R_i^l u_l^l. \tag{4.48}$$

In case ω_{il}^l, $\dot{\omega}_{il}^l$ are not available, we propose the following controller for the follower spacecraft:

$$\begin{aligned} u_f =& m_f \ddot{\rho}_d + C_f(\omega_{id}^l)\dot{\rho}_d + D_f(\omega_{id}^l, \dot{\bar{\omega}}_{id}^l)\rho \\ &+ n_f(r_l, \rho) - K_f(\dot{\rho}_0 - \dot{\rho}_r) \end{aligned} \tag{4.49}$$

$$\dot{\rho}_r = \dot{\rho}_d - \Lambda_f e_f \tag{4.50}$$

$$\dot{\rho}_0 = \dot{\hat{\rho}} - \Lambda_f \tilde{\rho}, \tag{4.51}$$

with $\dot{\rho}_r$ and $\dot{\rho}_0$ as defined in (4.42) and (4.43), together with the observer (4.44)-(4.45). Note that we have not used the actual orbital angular velocity and acceleration of the leader spacecraft, ω_{il}^l and $\dot{\omega}_{il}^l$, but the desired velocity and an estimate of the desired acceleration. The desired angular velocity ω_{id}^{l}, is a function of the desired position and velocity of the leader spacecraft, that is

$$\omega_{id}^{l}\left(\dot{r}_{d}, r_{d}\right) := R_{i}^{l} \frac{S\left(r_{d}^{i}\right) \dot{r}_{d}^{i}}{\left|r_{d}\right|^{2}} \tag{4.52}$$

The desired angular velocity is found by differentiating (4.52):

$$\begin{aligned} \dot{\omega}_{id}^{l} =& - R_{i}^{l} S(\omega_{il}^{l})\omega_{id}^{i} \\ &+ R_{i}^{l} \frac{\left\{S\left(r_{d}\right) \ddot{r}_{d}\right\} r_{d}^{\top} r_{d} - 2\left\{S\left(r_{d}\right) \dot{r}_{d}\right\} \dot{r}_{d}^{\top} r_{d}}{\left|r_{d}\right|^{4}} \end{aligned} \tag{4.53}$$

We see that this would require knowledge of the actual angular velocity of the leader spacecraft, ω_{il}^{l}, so we will instead use the estimate

$$\begin{aligned} \dot{\bar{\omega}}_{id}^{l} :=& \dot{\omega}_{id}^{l} + R_{i}^{l} S(\omega_{il}^{l})\omega_{id}^{i} \\ =& R_{i}^{l} \frac{\left\{S\left(r_{d}\right) \ddot{r}_{d}\right\} r_{d}^{\top} r_{d} - 2\left\{S\left(r_{d}\right) \dot{r}_{d}\right\} \dot{r}_{d}^{\top} r_{d}}{\left|r_{d}\right|^{4}}. \end{aligned} \tag{4.54}$$

For later reference we define

$$\dot{\bar{\omega}}_{ld}^{l} := \dot{\bar{\omega}}_{id}^{l} - \dot{\omega}_{il}^{l} \tag{4.55}$$

The system can be written on the same cascaded form $\dot{x} = f\left(t, x, \theta_{1}\right) + g\left(t, x, \theta\right)$, with f as in (4.46) and

$$g\left(t, x, \theta\right) = \begin{bmatrix} \frac{1}{m_{f}}\tilde{g} \\ 0 \\ \frac{1}{m_{f}}\tilde{g} \\ 0 \end{bmatrix}, \tag{4.56}$$

where

$$\begin{aligned} \tilde{g} :=& C_{f}(\omega_{id}^{l})\dot{\rho}_{d} - C_{f}(\omega_{il}^{l})\dot{\rho} + D_{f}(\omega_{id}^{l}, \dot{\bar{\omega}}_{id}^{l})\rho \\ &- D_{f}(\omega_{il}^{l}, \dot{\omega}_{il}^{l})\rho + d_{f}^{l} - \frac{m_{f}}{m_{l}} d_{l}^{l} - \frac{m_{f}}{m_{l}} R_{i}^{l} u_{l}^{l}. \end{aligned} \tag{4.57}$$

Proposition 4.4 *Under Assumption 4.5 and 4.6 the cascaded system*

$$\begin{aligned} \dot{x}_{1} &= f_{1}\left(t, x_{1}, \theta_{1}\right) + g\left(t, x, \theta\right) \\ \dot{x}_{2} &= f_{2}\left(t, x_{2}, \theta_{2}\right) \end{aligned}$$

with f_1 as in (4.46), f_2 as in (4.39) and g as in (4.47) is UGPES.

Proof. The proof is made by applying Theorem 2.5. We have already seen that UGPES of the driving subsystem can be shown by using

$$V_2(x_2) = x_2^\top W^\top R_2 W x_2$$

as a Lyapunov function. The conditions of Assumption 2.2 will also be satisfied with $\kappa_2 = k_l^\star/2$, $\underline{\kappa}_2 = 1/6m_l$ (independent of θ), $\overline{\kappa}_2(\delta) = 3k_l^\star(\delta_2)/\lambda_{\max}(\Lambda_l)$ and $V_{\delta_2} = V_2$, where $k_l^\star = 2m_l\lambda_{\max}(\Lambda_l) + 4m_l\beta_{d_l}/\delta_2$. To analyse the stability properties of the driven subsystem, we will use the following Lyapunov function candidate

$$V_1(x_1) := \frac{1}{2}x_1^\top W^\top R_1 W x_1$$

where $R_1 := \mathrm{diag}(m_f I_{3\times3}, 2K_f\Lambda_f^{-1} - m_f I_{3\times3}, m_f I_{3\times3}, 2K_f\Lambda_f^{-1}) \in \mathbb{R}^{12\times12}$. By defining the sliding variables:

$$t_1 := \dot{\rho} - \dot{\rho}_r$$
$$t_2 := \dot{\rho} - \dot{\rho}_0$$

such that $t_1 = \dot{e}_f + \Lambda_f e_f$ and $t_2 = \dot{\tilde{\rho}} + \Lambda_f\tilde{\rho}$, we find that the time derivative of the Lyapunov function is

$$\dot{V}_1 = m_f t_1^\top \dot{t}_1 + e_f^\top \Lambda_f \left(2K_f\Lambda_f^{-1} - m_f I_{3\times3}\right)\Lambda_f \dot{e}_f + m_f t_2^\top \dot{t}_2 + 2\tilde{p}^\top \Lambda_f K_f \dot{\tilde{p}}.$$

Inserting for the error dynamics (4.46) it can be shown that the time derivative of the Lyapunov function candidate can be compactly written as

$$\dot{V}_1 = -x_1^\top Q_f x_1 - t_2^\top (l_f m_f I_{3\times3} - 2K_f)\, t_2$$

where $Q_f := (K_f - m_f\Lambda_f, K_f, K_f, K_f) \in \mathbb{R}^{12\times12}$. For any $l_f > 2k_f/m_f$, and any $k_f > 2m_f\lambda_{\max}(\Lambda_f)$ we get that $\dot{V}_1 \le -\frac{1}{2}k_f|x_1|^2$. Hence, Assumption 2.1 holds with $V_{\delta_1} = V_1$, $\kappa_1 = k_f/2$, $\underline{\kappa}_1 = 1/6m_f$ (independent of θ) and $\overline{\kappa}_1 = 3k_f^\star/\lambda_{\max}(\Lambda_f)$. Furthermore, for $g(t,x,\theta_2) = (\bar{g}/m_f, 0, \bar{g}/m_f, 0)$ with $\bar{g}$ as in (4.48) we find that

$$\begin{aligned}
\frac{\partial V_1}{\partial x_1} g(t,x,\theta_2) &= (t_1 + t_2)^\top \bar{g} \\
&= (t_1 + t_2)^\top \left(d_f^l - \frac{m_f}{m_l}d_l^l - \frac{m_f}{m_l}R_i^l u_l^l\right) \\
&\le k\,|x_1|\,(1 + |\theta_2|\,|x_2|)
\end{aligned}$$

for some constant k independent of t, θ and x, where we have used Assumption 4.6 and 4.5, that

$$\begin{aligned}(t_1+t_2)^\top C_f(\omega_{il}^l)(\dot\rho-\frac{1}{2}(\dot\rho_0+\dot\rho_r)) &= (t_1+t_2)^\top C_f(\omega_{il}^l)(t_1+t_2)\\ &= 0,\end{aligned}$$

and that the controller in (4.34) can be bounded, in the sense that

$$|u_l(t,x_2)|\le m_l\beta_{\ddot r_d}+m_l\mu+2|\theta_2||x_2|. \tag{4.58}$$

Therefore, Assumption 2.3 of Theorem 2.5 holds, and the cascaded system is UGPES. ■

Proposition 4.5 *Under Assumption 4.5, 4.6 and 4.7 the cascaded system*

$$\begin{aligned}\dot x_1 &= f_1(t,x_1,\theta_1)+g(t,x,\theta)\\ \dot x_2 &= f_2(t,x_2,\theta_2)\end{aligned}$$

with f_1 as in (4.46), f_2 as in (4.39) and g as in (4.56) is UGPAS.

Proof. The proof is made by applying Theorem 2.7. Notice that Assumption 2.1 and 2.2 were shown to hold in the Proof of Theorem 4.4 with $\underline{\kappa}_1$ and $\underline{\kappa}_2$ independent of θ. Now,

$$\frac{\partial V_1}{\partial x_1}g(t,x,\theta_2)=(t_1+t_2)^\top\tilde g$$

with

$$\begin{aligned}\tilde g := {}& C_f(\omega_{id}^l)\dot\rho_d-C_f(\omega_{il}^l)\dot\rho+D_f(\omega_{id}^l,\dot\omega_{id}^l)\rho\\ &-D_f(\omega_{il}^l,\dot\omega_{il}^l)\rho+d_f^l-\frac{m_f}{m_l}d_l^l-\frac{m_f}{m_l}R_i^l u_l^l.\end{aligned}$$

By Assumption 4.5 and 4.7, and that $e_l=r_l-r_d$, and $e_f=\rho-\rho_d$, we have from (4.31) that

$$\begin{aligned}|C_f(\omega_{il}^l(t))\dot\rho(t)| &\le 2m_f|\omega_{il}||\dot\rho|\\ &\le 2m_f\frac{|r_l||\dot r_l|}{|r_l|^2}|\dot e_f+\dot\rho_d|\\ &\le 2m_f(|\dot e_l|+\beta_{\dot r_d})(|\dot e_f|+\beta_{\dot\rho_d})\\ &\le 2m_f(|x_2|+\beta_{\dot r_d})(|x_1|+\beta_{\dot\rho_d})\end{aligned}$$

By the same arguments, we find the bounds

$$\left|C_f(\omega_{id}^l(t))\dot{\rho}_d(t)\right| \leq 2m_f \frac{\beta_{r_d}\beta_{\dot{r}_d}}{\alpha_{r_d}^2}\beta_{\dot{\rho}_d}$$

and

$$\left|D_f(\omega_{id}^l(t),\dot{\omega}_{id}^l(t))\rho(t)\right| \leq m_f\left(4\frac{\beta_{r_d}^2\beta_{\dot{r}_d}^2+\beta_{r_d}^3\beta_{\ddot{r}_d}+2\beta_{r_d}^2\beta_{\dot{r}_d}^2}{\alpha_{r_d}^4}\right)\left(|x_1|+\beta_{\rho_d}\right).$$

From (4.39), $m_l(\ddot{r}_l-\ddot{r}_d)=-K_lYx_2+d_l$, and by Assumption 4.5, 4.7 and 4.6, we find that

$$\begin{aligned}\left|D_f(\omega_{il}^l(t),\dot{\omega}_{il}^l(t))\rho(t)\right| \leq\ & 3m_f\left(|x_2|+\beta_{\dot{r}_d}\right)^2\left(|x_1|+\beta_{\rho_d}\right)\\ &+\frac{m_f}{m_l}\left(2|\theta_2||x_2|+\beta_{d_l}\right)\left(|x_1|+\beta_{\rho_d}\right)\\ &+m_f\beta_{\ddot{r}_d}\left(|x_1|+\beta_{\rho_d}\right)\end{aligned}$$

Finally, with u_l bounded as in (4.58), $|d_l(t)|\leq\beta_{d_l}$ and $|d_f(t)|\leq\beta_{d_f}$ we find that there exist a constant γ, such that

$$\frac{\partial V_1}{\partial x_1}g(t,x,\theta_2)\leq\gamma|x_1|\left((|x_1|+1)|x_2||\theta_2|+|x_2|+|x_2|^2+1\right).$$

Therefore, Assumption 2.7 of Theorem 2.7 holds, and the cascaded system is UGPAS. ■

4.2.4 Simulation study

Let the reference trajectory of the leader spacecraft be an eccentric orbit with radius of perigee $R_p=10000000$ and radius of apogee $R_a=30000000$, which can be generated by numerical integration of

$$\ddot{r}_d=-m_l\frac{\mu}{|r_d|^3}r_d \tag{4.59}$$

with $r_d(0)=\mathrm{col}(R_p,0,0)$ and $\dot{r}_d(0)=\mathrm{col}(0,V_p,0)$, and where

$$V_p=\sqrt{2\mu\left(\frac{1}{R_p}-\frac{1}{(R_p+R_a)}\right)}.$$

Out of simplicity we have chosen the desired perigee to lie on the first axis of the inertial coordinate system. The initial values of the leader spacecraft are $r_l(0) = \mathrm{col}(R_p + 4, -3, 5)$ and $\dot{r}_l(0) = \dot{r}_d(0) + \mathrm{col}(0.4, -0.8, -0.2)$. The control parameters are $\Lambda_l = 0.04 I_{3\times 3}$, $k_l = 6$ and $l_l = 0.3$. These choices of control gains are chosen based on the outcome of the Lyapunov analysis. The initial values of the controller are chosen as the initial values of the reference trajectory, i.e. $\hat{r}(0) = r_d(0)$ and $a_l(0) = \dot{r}_d(0)$.

The reference trajectory of the follower spacecraft are chosen as the solutions of a special case of the Clohessy-Wiltshire equations, Clohessy and Wiltshire (1960). We use

$$p_d(t) = \begin{bmatrix} 10\cos\nu_d(t) \\ -20\sin\nu_d(t) \\ 0 \end{bmatrix} \tag{4.60}$$

and its time derivatives. Here, ν_d is the desired true anomaly, and where ν_d and $\dot{\nu}_d$ are found by

$$\ddot{\nu}_d(t) = \frac{-2\mu\left(1 + e_d\cos\nu_d(t)\right)^3\sin\nu_d(t)}{\left(R_p + R_a\right)^3\left(1 - e_d^2\right)^3}$$

numerical integration. With our choice of reference trajectory (4.59), $|\dot{\omega}_{id}^i| = \ddot{\nu}_d^i$. Since the reference trajectory is chosen to start at perigee, $\nu_d(0) = 0$, and $\dot{\nu}_d(0) = V_p/R_p$. The eccentricity of the reference trajectory is constant, and can be calculated from R_a and R_p to be $e_d = 0.5$. This choice of reference trajectory means that the two spacecraft are in the same orbital plane, and that the follower spacecraft will "evolve" around the leader spacecraft as the two spacecraft orbit Earth. The initial values of the follower spacecraft are $p(0) = \mathrm{col}(5, -7, 3)$ and $\dot{p}(0) = \mathrm{col}(-0.3, 0.2, 0.6)$. The control parameters are the same as for the leader spacecraft, i.e. $\Lambda_f = \Lambda_l$, $k_f = k_l$ and $l_f = l_l$. The initial parameters of the controller are chosen to be $\hat{p}(0) = p_d(0) = \mathrm{col}(10, 0, 0)$ and $a_f(0) = \dot{p}_d(0) = \mathrm{col}(0, 0, 0)$. We use $m_f = m_l = 100$ kg both in the model and the control structure. Both spacecraft are subject to J_2 perturbations, as described in Section 3.3.1, that is:

$$d_f^i = \frac{1}{2}J_2 R_e^2 m_f \mu \begin{bmatrix} 15\frac{r_{f,1}r_{f,3}^2}{|r_f|^7} - 3\frac{r_{f,1}}{|r_f|^5} \\ 15\frac{r_{f,2}r_{f,3}^2}{|r_f|^7} - 3\frac{r_{f,2}}{|r_f|^5} \\ 15\frac{r_{f,3}^3}{|r_f|^7} - 9\frac{r_{f,3}}{|r_f|^5} \end{bmatrix},$$

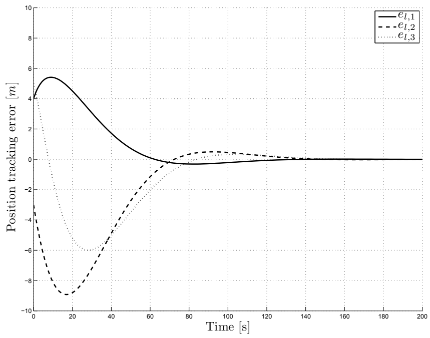

Figure 4.4: Position tracking error of leader spacecraft

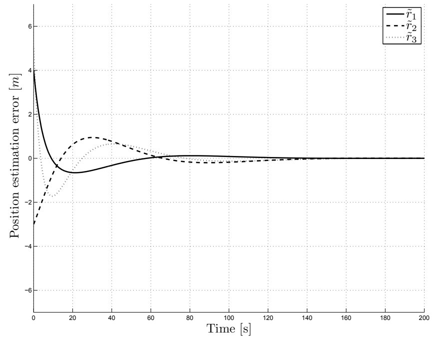

Figure 4.5: Position estimation error of leader spacecraft

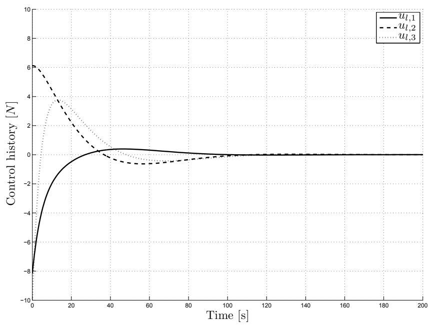

Figure 4.6: Control forces acting on leader spacecraft

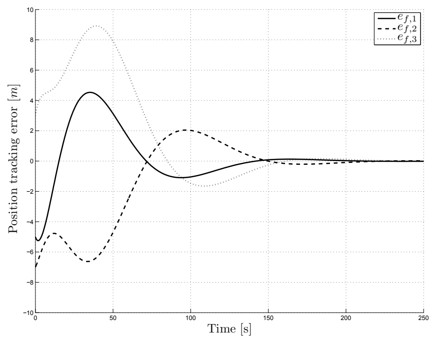

Figure 4.7: Position tracking error of follower spacecraft

$$d_l^i = \frac{1}{2} J_2 R_e^2 m_l \mu \begin{bmatrix} 15\frac{r_{l,1} r_{l,3}^2}{|r_l|^7} - 3\frac{r_{l,1}}{|r_l|^5} \\ 15\frac{r_{l,2} r_{l,3}^2}{|r_l|^7} - 3\frac{r_{l,2}}{|r_l|^5} \\ 15\frac{r_{l,3}^3}{|r_l|^7} - 9\frac{r_{l,3}}{|r_l|^5} \end{bmatrix},$$

where R_e is the mean equatorial radius of the Earth, J_2 is the second zonal harmonics, $\mathrm{col}(r_{f,1}, r_{f,2}, r_{f,3}) =: r_f^i$ and $\mathrm{col}(r_{l,1}, r_{l,2}, r_{l,3}) =: r_l^i$.

Figure 4.4, 4.5 and 4.6 shows the position tracking error, position estimation error and control history of the leader spacecraft, where as Figure 4.7, 4.8 and 4.9 are the equivalent figures for the follower spacecraft. No optimization has been used in picking the control gains, but they are chosen such that the actuation is kept 'moderate'. This comes at the cost of a longer settling time. It is clear that better initial conditions would improve

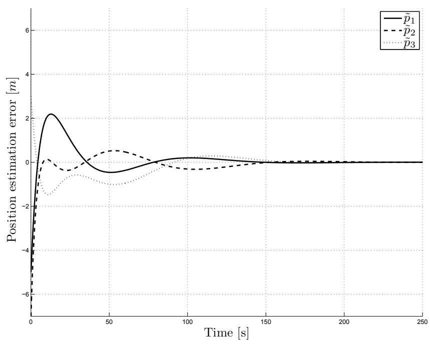

Figure 4.8: Position estimation error of follower spacecraft

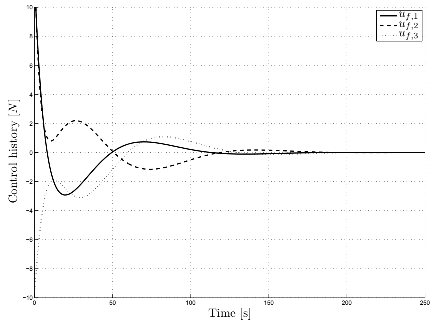

Figure 4.9: Control forces acting on follower spacecraft

the overshoot and the settling time significantly. By comparing the figures for the performance of the leader spacecraft with those for the follower spacecraft, we see that the follower spacecraft has a longer settling time. This is natural since the same control gains have been used for both spacecraft, and since control forces acting on the leader spacecraft influences the follower spacecraft model.

Chapter 5

Output tracking control of leader-follower formation: rotational case

This chapter is based on Grøtli and Gravdahl (2008b).

5.1 Models and basic assumptions

We will use the models from Section 3.2, restated here for the sake of completeness. The model for the leader spacecraft is

$$\dot{q}_{il} = \frac{1}{2}\begin{bmatrix} -\epsilon_{il}^{\top} \\ E\left(q_{il}\right) \end{bmatrix} \omega_{il}^{l}, \tag{5.1}$$

$$J_l\dot{\omega}_{il}^{l} + C_l(\omega_{il}^{l})\omega_{il}^{l} = \tau_l + d_l, \tag{5.2}$$

with $J_l \in \mathbb{R}^{3\times 3}$ being the leader spacecraft inertia matrix, ω_{il}^{l} the angular velocity of the spacecraft relative to the inertial frame, $C_l\left(\omega_{il}^{l}\right) = -S\left(J_l\omega_{il}^{l}\right)$ and τ_l and d_l the input and disturbance moments on the leader spacecraft, respectively.

The model for the follower spacecraft is

$$\dot{q}_{lf} = \frac{1}{2}\begin{bmatrix} -\epsilon_{lf}^{\top} \\ E\left(q_{lf}\right) \end{bmatrix} \omega_{lf}^{f}, \tag{5.3}$$

$$J_f\dot{\omega}_{lf}^{f} + C_f(\omega_{lf}^{f})\omega_{lf}^{f} + n_f(\omega_{il}^{l}, \omega_{lf}^{f}) = \Gamma_a + \Gamma_d, \tag{5.4}$$

with $J_f \in \mathbb{R}^{3\times 3}$ being the follower spacecraft inertia matrix, $\omega_{lf}^f = \omega_{if}^f - R_l^f \omega_{il}^l$ the angular velocity of the follower spacecraft relative to the leader spacecraft, $C_f(\omega_{lf}^f) = -S(J_f \omega_{lf}^f)$ and

$$\begin{aligned} n_f =& (S(R_l^f \omega_{il}^l) J_f R_l^f - J_f R_l^f J_l^{-1} S(\omega_{il}^l) J_l) \omega_{il}^l \\ &+ (J_f S(R_l^f \omega_{il}^l) - S(J_f R_l^f \omega_{il}^l) + S(R_l^f \omega_{il}^l) J_f) \omega_{lf}^f. \end{aligned} \tag{5.5}$$

Furthermore,

$$\Gamma_a = \tau_f^f - J_f R_l^f J_l^{-1} \tau_l^l, \tag{5.6}$$

and

$$\Gamma_d = d_f^f - J_f R_l^f J_l^{-1} d_l^l, \tag{5.7}$$

with τ_f and d_f as the input and disturbance moments on the follower spacecraft, respectively.

We pose the following assumption on the spacecraft models:

Assumption 5.1 *The inertia matrices J_i, $i \in \{l, f\}$ are symmetric and positive definite, and satisfy the inequalities*

$$\alpha_{J_i} \leq |J_i| \leq \beta_{J_i},$$

with $\alpha_{J_i}, \beta_{J_i} \in \mathbb{R}$ being positive constants.

Assumption 5.2 *The disturbance moments d_i, $i \in \{l, f\}$ are bounded as*

$$|d_i(t)| \leq \beta_{d_i},$$

with $\beta_{d_i} \in \mathbb{R}$ being positive constants.

5.2 Controller-observer design

5.2.1 Leader spacecraft

The desired angular velocity of the leader spacecraft is usually given with reference to the inertial frame as ω_{id}^i. In the leader spacecraft frame, it is

$$\omega_{id}^l = R_i^l \omega_{id}^i,$$

where as its time derivative is

$$\begin{aligned} \dot{\omega}_{id}^l &= \dot{R}_i^l \omega_{id}^i + R_i^l \dot{\omega}_{id}^i \\ &= -S(\omega_{il}^l) \omega_{id}^l + R_i^l \dot{\omega}_{id}^i. \end{aligned}$$

We see that to evaluate the derivative we need to know the actual velocity of the leader spacecraft ω_{il}^l, so we will therefore use the modified acceleration vector

$$\begin{aligned} a_d &= -S(\omega_{id}^l)\omega_{id}^l + R_i^l \dot{\omega}_{id}^i \\ &= R_i^l \dot{\omega}_{id}^i. \end{aligned} \tag{5.8}$$

Let us assume the following:

Assumption 5.3 *The desired angular velocity and the desired angular acceleration of the leader spacecraft are bounded, i.e.* $\left|\omega_{id}^l(t)\right| \leq \beta_{\omega_{id}^l}$ *and* $\left|\dot{\omega}_{id}^l(t)\right| \leq \beta_{\dot{\omega}_{id}^l}$ *for all* $t \geq t_0 \geq 0$*, for some positive constants* $\beta_{\omega_{id}^l}$ *and* $\beta_{\dot{\omega}_{id}^l}$.

The following controller-observer scheme is the same as in (Caccavale and Villani, 1999, Theorem 1). Let the control law be

$$\tau_l^l = J_l a_r + C_l(\omega_o)\omega_r + k_v(\omega_r - \omega_o) - k_p \epsilon_{dl} \tag{5.9}$$

$$a_r = a_d - \frac{1}{2}\lambda_d E(q_{de})\omega_{de} \tag{5.10}$$

$$\omega_r = \omega_{id}^l - \lambda_d \epsilon_{de} \tag{5.11}$$

$$\omega_o = \omega_{ie}^l - \lambda_e \epsilon_{el}, \tag{5.12}$$

with $k_v, k_p, \lambda_e, \lambda_d \in \mathbb{R}$ constants to be defined, ϵ_{dl} as the vector part of the quaternion product $q_{dl} = \bar{q}_{id} \otimes q_{il}$, ϵ_{de} as the vector part of $q_{de} = \bar{q}_{id} \otimes q_{ie}$, ϵ_{el} and η_{el} as the vector and scalar part of $q_{el} = \bar{q}_{ie} \otimes q_{il}$, respectively, $\omega_{de}^l = \omega_{ie}^l - \omega_{id}^l$ and $E(q_{de}) = \eta_{de} I + S(\epsilon_{de})$. Here, q_{id} represents the orientation of the desired frame, q_{ie} the orientation of the estimated frame, and finally q_{il} the actual orientation of the leader spacecraft, all relative to the inertial frame. Let the observer be

$$\dot{z} = a_r + J_l^{-1}(l_p \epsilon_{el} - k_p \epsilon_{dl} + l_v \lambda_e \eta_{el} \epsilon_{el}) \tag{5.13}$$

$$\omega_{ie}^l = z + \lambda_e \epsilon_{el} + 2J_l^{-1} l_v \epsilon_{el}, \tag{5.14}$$

with $l_v, l_p \in \mathbb{R}$ constants to be defined.

Let us first define the sliding variables

$$\sigma_d = \omega_{il}^l - \omega_r \tag{5.15}$$

$$= \omega_{dl}^l + \lambda_d \epsilon_{de}, \tag{5.16}$$

and

$$\sigma_e = \omega_{il}^l - \omega_o \tag{5.17}$$
$$= \omega_{el}^l + \lambda_e \epsilon_{el}. \tag{5.18}$$

Define $\tilde{\eta}_{dl} := 1 - \eta_{dl}$ and $\tilde{\eta}_{el} := 1 - \eta_{el}$. Let $x_2 := \mathrm{col}(\sigma_d, \tilde{\eta}_{dl}, \epsilon_{dl}, \sigma_e, \tilde{\eta}_{el}, \epsilon_{el})$ and let $\theta_2 := \mathrm{col}\,(k_v, l_v)$. The error dynamics can be written on state space form $\dot{x}_2 = f_2\,(t, x_2, \theta_2)$, where

$$f_2\,(t, x_2, \theta_2) = \begin{bmatrix} J_l^{-1}\xi_3 \\ \frac{1}{2}\begin{bmatrix} \epsilon_{dl}^\top \\ E\,(q_{dl}) \end{bmatrix}\omega_{dl}^l \\ J_l^{-1}\xi_4 \\ \frac{1}{2}\begin{bmatrix} \epsilon_{el}^\top \\ E\,(q_{el}) \end{bmatrix}\omega_{el}^l \end{bmatrix} \tag{5.19}$$

with

$$\begin{aligned} \xi_3 = &- C_l(\omega_{il}^l)\sigma_d - k_v\sigma_d - k_p\epsilon_{dl} + k_v\sigma_e - C_l(\sigma_e)\omega_r \\ &- J_l S(\omega_{ld}^l)\omega_{id}^l + d_l \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} \xi_4 = &- (l_v E(q_{el}) - k_v I)\sigma_e - l_p\epsilon_{el} - k_v\sigma_d - C_l(\sigma_e)\omega_r \\ &- C_l(\omega_{il}^l)\sigma_d + d_l. \end{aligned} \tag{5.21}$$

Remark 5.1 *Note that we have chosen to characterize perfect tracking in terms of the quaternion error to when $\eta_{dl} = +1$ and $\eta_{el} = +1$, cf. the discussion about perfect tracking in Section 1.3.2. We could just as well have used $\eta_{dl} = -1$ and $\eta_{el} = -1$, or both - that is, defined tracking error in terms of the scalar part of the quaternion product as $1 - |\eta_{dl}|$ and $1 - |\eta_{el}|$. Throughout the literature it has been common to use the signum function in the control law for efficient maneuvers. Such an approach would not fit our framework, because of the discontinuities it would introduce. A thorough analysis of stability with respect to sets using discontinuous Lyapunov functions can be found in Fragopoulos and Innocenti (2004). In Kristiansen et al. (2009) stability of both equilibrium positions are shown.*

Proposition 5.1 *Let Assumption 5.1, 5.3 and 5.2 hold. Then, the system $\dot{x}_2 = f_2\,(t, x_2, \theta_2)$ is UPES.*

Proof. The proof is mostly similar to the proof of (Caccavale and Villani, 1999, Theorem 1). Consider the positive definite Lyapunov function candidate

$$\begin{aligned} V_2 =& \frac{1}{2}\sigma_d^\top J_l \sigma_d + k_p((1-\eta_{dl})^2 + \epsilon_{dl}^\top \epsilon_{dl}) \\ &+ \frac{1}{2}\sigma_e^\top J_l \sigma_e + l_p((1-\eta_{el})^2 + \epsilon_{el}^\top \epsilon_{el}). \end{aligned} \tag{5.22}$$

Following the steps of the proof of (Caccavale and Villani, 1999, Theorem 1) we find that the time derivative of the Lyapunov function candidate along the error dynamics are

$$\begin{aligned} \dot{V}_2 =& - k_v \sigma_d^\top \sigma_d - k_p \lambda_d \eta_{el} \epsilon_{dl}^\top \epsilon_{dl} + k_p \lambda_d \eta_{dl} \epsilon_{el}^\top \epsilon_{dl} \\ &- \sigma_d^\top C_l(\sigma_e)\omega_r - \sigma_d^\top J_l S(\omega_{ld}^l)\omega_{id}^l + \sigma_d^\top d_l \\ &- (l_v \eta_{el} - k_v)\sigma_e^\top \sigma_e - l_p \lambda_e \epsilon_{el}^\top \epsilon_{el} \\ &- \sigma_e^\top C_l(\sigma_e)\omega_r - \sigma_e^\top C_l(\omega_{il}^l)\sigma_d + \sigma_e^\top d_l. \end{aligned}$$

From Remark 3.4, that is, since the matrix $C_l(\cdot)$ is linear in its argument, we have that

$$|C_l(a)b| \leq \beta_{J_l} |a| |b| . \tag{5.23}$$

By (5.23), Young's inequality and (5.11) we have that

$$\sigma_d^\top C_l(\sigma_e)\omega_r \leq \frac{1}{2}\beta_{J_l}(|\sigma_d|^2 + |\sigma_e|^2)(\beta_{\omega_{id}^l} + \lambda_d |\epsilon_{de}|). \tag{5.24}$$

By (5.23), Young's inequality, (5.11) and (5.15), we have that

$$\sigma_e^\top C_l(\omega_{il}^l)\sigma_d \leq \frac{1}{2}\beta_{J_l}(|\sigma_d|^2 + |\sigma_e|^2)(|\sigma_d| + \lambda_d |\epsilon_{de}| + \beta_{\omega_{id}^l}). \tag{5.25}$$

By (5.23) and (5.11) we have that

$$\sigma_e^\top C_l(\sigma_e)\omega_r \leq \beta_{J_l} |\sigma_e|^2 (\lambda_d |\epsilon_{de}| + \beta_{\omega_{id}^l}). \tag{5.26}$$

By (5.23), (5.15) and that $|\epsilon_{de}| \leq |\epsilon_{el}| + |\epsilon_{dl}|$ we have that

$$\begin{aligned} \sigma_d^\top J_l S(\omega_{ld}^l)\omega_{id}^l &\leq \beta_{J_l}\beta_{\omega_{id}^l} |\sigma_d| (|\sigma_d| + \lambda_d |\epsilon_{de}|) \\ &\leq \beta_{J_l}\beta_{\omega_{id}^l} (|\sigma_d|^2 + \lambda_d |\sigma_d| (|\epsilon_{el}| + |\epsilon_{dl}|)) \\ &\leq \beta_{J_l}\beta_{\omega_{id}^l} ((1 + \frac{\lambda_d}{2}) |\sigma_d|^2 + \frac{\lambda_d}{2}(|\epsilon_{el}|^2 + |\epsilon_{dl}|^2)). \end{aligned} \tag{5.27}$$

Inserting for the bounds (5.24-5.27), we get that

$$
\begin{aligned}
\dot{V}_2 \leq & -\left(k_v - \beta_{J_l}(\lambda_d\,|\epsilon_{de}| + \frac{1}{2}\,|\sigma_d| + (2+\frac{\lambda_d}{2})\beta_{\omega^l_{id}})\right)|\sigma_d|^2 \\
& - \left(l_v\eta_{el} - k_v - 2\beta_{J_l}(\lambda_d\,|\epsilon_{de}| + \beta_{\omega^l_{id}} + \frac{1}{4}\,|\sigma_d|)\right)|\sigma_e|^2 \\
& - \frac{1}{2}(k_p\lambda_d\eta_{el} - \beta_{J_l}\beta_{\omega^l_{id}}\lambda_d)\,|\epsilon_{dl}|^2 \\
& - \frac{1}{2}(l_p\lambda_e - \beta_{J_l}\beta_{\omega^l_{id}}\lambda_d)\,|\epsilon_{el}|^2 \\
& - \frac{1}{2}\begin{bmatrix}|\epsilon_{dl}| \\ |\epsilon_{el}|\end{bmatrix}^\top \begin{bmatrix} k_p\lambda_d\eta_{el} & -k_p\lambda_d \\ -k_p\lambda_d & l_p\lambda_e\end{bmatrix}\begin{bmatrix}|\epsilon_{dl}| \\ |\epsilon_{el}|\end{bmatrix} \\
& + (|\sigma_d| + |\sigma_e|)\beta_{d_l}.
\end{aligned}
$$

For any $|x_2| \leq \bar{\Delta}_2 < 1$, we have that $\eta_{el} \geq \sqrt{1-\bar{\Delta}_2} > 0$. For any $\delta_2 \leq |x_2|$, we define

$$
\begin{aligned}
k_v^\star &:= \beta_{J_l}\left(\lambda_d + \frac{1}{2}\bar{\Delta}_2 + (2+\frac{\lambda_d}{2})\beta_{\omega^l_{id}} + \frac{\beta_{d_l}}{\delta_2}\right), \\
l_v^\star &:= \frac{1}{\sqrt{1-\bar{\Delta}_2}}\left(k_v + 2\beta_{J_l}\left(\lambda_d + \frac{1}{2}\bar{\Delta}_2 + \beta_{\omega^l_{id}}\right) + \frac{\beta_{d_l}}{\delta_2}\right), \\
k_p^\star &:= \frac{\beta_{J_l}\beta_{\omega^l_{id}}}{\sqrt{1-\bar{\Delta}_2}}, \\
l_p^\star &:= \max\left\{\frac{\beta_{J_l}\beta_{\omega^l_{id}}\lambda_d}{\lambda_e}, \frac{k_p\lambda_d}{\lambda_e\sqrt{1-\bar{\Delta}_2}}\right\},
\end{aligned}
$$

such that with $k_v > k_v^\star$, $l_v > l_v^\star(k_v)$, $k_p > k_p^\star$ and $l_p > l_p^\star(k_p)$ condition (2.15) of Theorem 2.4 is satisfied, provided that η_{el} does not change sign. Note that for the considered domain of the state space, namely where $|x_2| \leq \bar{\Delta}_2$, V_2 is in fact a proper Lyapunov function, i.e. its time derivative can be bounded as in (2.15). To see this, let c_1 and c_2 be positive constants. For $|x_2| \leq \bar{\Delta}_2$ we have that $\eta_{el}, \eta_{dl} > 0$, so $-c_1|\epsilon_{dl}|^2 \leq -1/2c_1(|\epsilon_{dl}|^2 + (1-\eta_{dl})^2)$ and $-c_2|\epsilon_{el}|^2 \leq -1/2c_2(|\epsilon_{el}|^2 + (1-\eta_{el})^2)$. Condition (2.14) is satisfied with $V_\delta = V_2$, $\underline{\kappa}(\delta) = \min\{1/2\alpha_{J_l}, k_p, l_p\}$, $\overline{\kappa}(\delta) = \max\{1/2\beta_{J_l}, 2k_p, 2l_p\}$. Hence, for any $x(0) \in \mathcal{B}_{\Delta_2}$, where $\Delta_2 := \sqrt{\overline{\kappa}(\delta)/\underline{\kappa}(\delta)}\bar{\Delta}_2$, we are ensured that η_{el} does not change sign. Furthermore,

$$
\lim_{\delta_2\to 0}\frac{\overline{\kappa}(\delta_2)\,\delta_2^p}{\underline{\kappa}(\delta_2)} = \lim_{\delta_2\to 0}\frac{\max\{\frac{1}{2}\beta_{J_l}, 2k_p, 2l_p\}\delta_2^2}{\min\{\frac{1}{2}\alpha_{J_l}, k_p, l_p\}} = 0,
$$

and we can conclude UPES with $\theta = \mathrm{col}(k_v, l_v)$ as tuning parameter. ■

5.2.2 Follower spacecraft

In the design and analysis of the follower spacecraft, we will *overline* the subscripts to distinguish vectors from the vectors related to the leader spacecraft. The subscript $\bar{d}$ denote the desired frame and $\bar{e}$ the estimated frame of follower spacecraft. E.g. $\omega^i_{l\bar{d}}$ will be the desired angular velocity of the follower spacecraft relative to the leader spacecraft.

Consider the control law:

$$
\begin{aligned}
\tau_f^f &= J_f a_{\bar{r}} + C_f(\omega_{\bar{o}})\omega_{\bar{r}} + k_{\bar{v}}(\omega_{\bar{r}} - \omega_{\bar{o}}) - k_{\bar{p}}\epsilon_{\bar{d}f} && (5.28)\\
a_{\bar{r}} &= a_{\bar{d}} - \frac{1}{2}\lambda_{\bar{d}} E(q_{\bar{d}\bar{e}})\omega^f_{\bar{d}\bar{e}} && (5.29)\\
\omega_{\bar{r}} &= \omega^f_{l\bar{d}} - \lambda_{\bar{d}}\epsilon_{\bar{d}\bar{e}} && (5.30)\\
\omega_{\bar{o}} &= \omega^f_{l\bar{e}} - \lambda_{\bar{e}}\epsilon_{\bar{e}f}, && (5.31)
\end{aligned}
$$

with $k_{\bar{v}}, k_{\bar{p}}, \lambda_{\bar{d}}, \lambda_{\bar{e}} \in \mathbb{R}$ positive constants, $\epsilon_{\bar{d}f}$ as the vector part of the quaternion product $q_{\bar{d}f} = \bar{q}_{l\bar{d}} \otimes q_{lf}$, $\epsilon_{\bar{d}\bar{e}}$ as the vector part of $q_{\bar{d}\bar{e}} = \bar{q}_{l\bar{d}} \otimes q_{l\bar{e}}$, $\epsilon_{\bar{e}f}$ as the vector part of $q_{\bar{e}f} = \bar{q}_{l\bar{e}} \otimes q_{lf}$, $\omega^f_{\bar{d}\bar{e}} = \omega^f_{l\bar{e}} - \omega^l_{l\bar{d}}$ and $E(q_{\bar{d}\bar{e}}) = \eta_{\bar{d}\bar{e}} I + S(\epsilon_{\bar{d}\bar{e}})$. Here, the desired orientation of the follower spacecraft relative to the leader is described by $q_{l\bar{d}}$, the actual orientation of the follower spacecraft relative to the leader is q_{lf}, and finally $q_{l\bar{e}}$ is the estimated orientation of the follower spacecraft relative to the leader. Since the states ω^f_{lf} and ω^l_{il} are assumed unknown, we have introduced the acceleration vector $a_{\bar{d}} = R^f_i \dot{\omega}^i_{l\bar{d}}$. Let the observer be

$$
\begin{aligned}
\dot{z} &= a_{\bar{r}} + J_f^{-1}(l_p\epsilon_{\bar{e}f} - k_p\epsilon_{\bar{d}f} + l_{\bar{v}}\lambda_{\bar{e}}\eta_{\bar{e}f}\epsilon_{\bar{e}f}) && (5.32)\\
\omega^f_{l\bar{e}} &= z + \lambda_{\bar{e}}\epsilon_{\bar{e}f} + 2J_f^{-1} l_{\bar{v}}\epsilon_{\bar{e}f}, && (5.33)
\end{aligned}
$$

with $l_{\bar{v}}$ and $l_{\bar{p}}$ positive constants.

To ease the analysis we will define the variables

$$
\begin{aligned}
\sigma_{\bar{d}} &= \omega^f_{lf} - \omega_{\bar{r}} && (5.34)\\
&= \omega^f_{\bar{d}f} + \lambda_{\bar{d}}\epsilon_{\bar{d}\bar{e}} && (5.35)
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{\bar{e}} &= \omega^f_{lf} - \omega_{\bar{o}} && (5.36)\\
&= \omega^f_{\bar{e}f} + \lambda_{\bar{e}}\epsilon_{\bar{e}f}. && (5.37)
\end{aligned}
$$

Define $\tilde{\eta}_{\bar{d}f} := 1-\eta_{\bar{d}f}$ and $\tilde{\eta}_{\bar{e}f} := 1-\eta_{\bar{e}f}$. Let $x_1 := \mathrm{col}(\sigma_{\bar{d}}, \tilde{\eta}_{\bar{d}f}, \epsilon_{\bar{d}f}, \sigma_{\bar{e}}, \tilde{\eta}_{\bar{e}f}, \epsilon_{\bar{e}f})$ and let $\theta_1 = \mathrm{col}\,(k_{\bar{v}}, l_{\bar{v}})$. We can write the error dynamics on state space form, as:

$$\begin{aligned} \dot{x}_1 &= f_1\,(t, x_1, \theta_1) + g\,(t, x, \theta) && (5.38)\\ \dot{x}_2 &= f_2\,(t, x_2, \theta_2)\,, && (5.39) \end{aligned}$$

where

$$f_1\,(t, x_1, \theta_1) := \begin{bmatrix} J_f^{-1}\xi_1 \\ \frac{1}{2}\begin{bmatrix} \epsilon_{\bar{d}f}^\top \\ E\left(q_{\bar{d}f}\right) \end{bmatrix} \omega_{\bar{d}f}^f \\ J_f^{-1}\xi_2 \\ \frac{1}{2}\begin{bmatrix} \epsilon_{\bar{e}f}^\top \\ E\left(q_{\bar{e}f}\right) \end{bmatrix} \omega_{\bar{e}f}^f \end{bmatrix} \qquad (5.40)$$

with

$$\begin{aligned} \xi_1 =& - C_f(\omega_{lf}^f)\sigma_{\bar{d}} - k_{\bar{v}}\sigma_{\bar{d}} - k_{\bar{p}}\epsilon_{\bar{d}f} + k_{\bar{v}}\sigma_{\bar{e}} - C_f(\sigma_{\bar{e}})\omega_{\bar{r}} \\ &+ J_f S(\omega_{lf}^f)\omega_{l\bar{d}}^f + d_f^f - J_f R_l^f J_l^{-1} d_l^l, \end{aligned}$$

$$\begin{aligned} \xi_2 =& - (l_{\bar{v}}E(q_{\bar{e}f}) - k_{\bar{v}}I)\sigma_{\bar{e}} - l_p\epsilon_{\bar{e}f} - k_{\bar{v}}\sigma_{\bar{d}} - C_f(\sigma_{\bar{e}})\omega_{\bar{r}} \\ &- C_f(\omega_{lf}^f)\sigma_{\bar{d}} + d_f^f - J_f R_l^f J_l^{-1} d_l^l, \end{aligned}$$

and

$$g\,(t, x, \theta) := \begin{bmatrix} -n_f(\omega_{il}^l, \omega_{lf}^f) - J_f S(\omega_{l\bar{d}}^f) R_l^f \omega_{il}^l - J_f R_l^f J_l^{-1}\tau_l^l \\ 0 \\ -n_f(\omega_{il}^l, \omega_{lf}^f) - J_f R_l^f J_l^{-1}\tau_l^l \\ 0 \end{bmatrix}. \qquad (5.41)$$

Finally, $f_2(t, x_2, \theta_2)$ is as in (5.19-5.21). We are now ready to state the following proposition:

Proposition 5.2 *Let Assumption 5.1 and 5.3 hold. Then the system (5.38-5.39) is UPES.*

Proof. To prove this proposition we will apply Theorem 2.6. We will first prove Assumption 2.4. Consider the positive-definite Lyapunov function

$$\begin{aligned} V_1 &= \frac{1}{2}\sigma_{\bar{d}}^\top J_f \sigma_{\bar{d}} + k_{\bar{p}}((1-\eta_{\bar{d}f})^2 + \epsilon_{\bar{d}f}^\top \epsilon_{\bar{d}f}) \\ &\quad + \frac{1}{2}\sigma_{\bar{e}}^\top J_f \sigma_{\bar{e}} + l_{\bar{p}}((1-\eta_{\bar{e}f})^2 + \epsilon_{\bar{e}f}^\top \epsilon_{\bar{e}f}) \end{aligned}$$

This function satisfies condition (2.23) of Theorem 2.6 with $V_{\delta_1} = V_1$, $\underline{\kappa}_1 = \min\ \{1/2\alpha_{J_f}, k_{\bar{p}}, l_{\bar{p}}\}$ and $\overline{\kappa}_1 = \max\ \{1/2\beta_{J_f}, 2k_{\bar{p}}, 2l_{\bar{p}}\}$. Notice that $\underline{\kappa}_1$ and $\overline{\kappa}_2$ are independent of θ. We will now prove Assumption 2.6, i.e. boundedness of the gradient of V_{δ_1} along the interconnection term $g(t,x,\theta)$. First we find bounds on the terms of τ_l, as stated in (5.9). From (5.10), (5.8), (5.11), (5.12), (5.15) and (5.17), and using that $\omega^l_{de} = \omega^l_{ie} - \omega^l_{id}$, we find that

$$|J_l a_r| \leq \beta_{J_l}\beta_{\dot{\omega}_{id}} + \frac{1}{2}\lambda_d \left(|\eta_{de}| + |\epsilon_{de}|\right)\left(|\sigma_d| + \lambda_d\,|\epsilon_{de}| + |\sigma_e| + \lambda_e\,|\epsilon_{el}|\right).$$

From (5.11), (5.15) and (5.17), we have that

$$|C_l\,(\omega_o)\,\omega_r| \leq \beta_{J_l}\left(|\sigma_d| + |\sigma_e| + \beta_{\omega_{id}} + \lambda_d\,|\epsilon_{de}|\right)\left(\beta_{\omega_{id}} + \lambda_d\,|\epsilon_{de}|\right).$$

Furthermore, from (5.5), and using that $|S\,(\alpha)\,| = |\alpha|$ and $|R| = 1$,

$$\begin{aligned}\left|n_f(\omega^l_{il}, \omega^f_{lf})\right| \;\leq\;& 2\beta_{J_f}\left(|\sigma_d| + \beta_{\omega_{id}} + \lambda_d\,|\epsilon_{de}|\right)^2 \\ &+3\beta_{J_f}\left(|\sigma_d| + \beta_{\omega_{id}} + \lambda_d\,|\epsilon_{de}|\right)\Big(|\sigma_{\bar{d}}| + \beta_{\omega_{l\bar{d}}} + \lambda_{\bar{d}}\,|\epsilon_{\bar{d}\bar{e}}|\Big).\end{aligned}$$

Notice that,

$$\frac{\partial V_1}{\partial x_1} = x_1^\top Q,$$

where $Q := \mathrm{diag}(J_f, 2k_{\bar{p}}I_{4\times4}, J_f, 2l_{\bar{p}}I_{4\times4})$, such that Q is independent of θ. By the above derived bounds, and using that $|\sigma_d| \leq |x_2|$, $|\sigma_e| \leq |x_2|$, $|\sigma_{\bar{d}}| \leq |x_1|$, $|\eta_{de}| \leq 1$, $|\epsilon_{de}| \leq 1$ and $|\epsilon_{\bar{d}\bar{e}}| \leq 1$, we have that the following holds:

$$\frac{\partial V_1}{\partial x_1}g(t,x,\theta) \leq a_0\,|x_1| + (a_1 + |\theta_2|)\,|x_1|\,|x_2| + a_2\,|x_1|\,|x_2|^2, \tag{5.42}$$

with a_0, a_1, a_2 being positive constants, independent of x_1, x_2, θ_1, θ_2 and t. For any $|x_2| \in \Delta_2$, with Δ_2 as in the proof of Proposition 5.1, Assumption 2.6 holds. The time derivative of V_1 along $\dot{x}_1 = f_1\,(t, x_1, \theta_1)$, is given by

$$\begin{aligned}\dot{V}_1 =& -k_{\bar{v}}\sigma^T_{\bar{d}}\sigma_{\bar{d}} - k_{\bar{p}}\lambda_{\bar{d}}\eta_{\bar{e}f}\epsilon^T_{\bar{d}f}\epsilon_{\bar{d}f} + k_{\bar{p}}\lambda_{\bar{d}}\eta_{\bar{d}f}\epsilon^T_{\bar{e}f}\epsilon_{\bar{d}f} \\ &- \sigma^T_{\bar{d}}C_f(\sigma_{\bar{e}})\omega_{\bar{r}} + \sigma^T_{\bar{d}}J_f S(\omega^f_{lf})\omega^f_{l\bar{d}} \\ &- (l_{\bar{v}}\eta_{\bar{e}f} - k_{\bar{v}})\sigma^T_{\bar{e}}\sigma_{\bar{e}} - l_{\bar{p}}\lambda_{\bar{e}}\epsilon^T_{\bar{e}f}\epsilon_{\bar{e}f} - \sigma^T_{\bar{e}}C_f(\sigma_{\bar{e}})\omega_{\bar{r}} \\ &- \sigma^T_{e}C_f(\omega^f_{lf})\sigma_{\bar{d}} + \left(\sigma^T_{\bar{d}} + \sigma^T_{\bar{e}}\right)d_f\end{aligned}$$

By deriving similar bounds as in (5.23-5.27), the time derivative of the Lyapunov function can be upper bounded by:

$$
\begin{aligned}
\dot{V}_1 \leq\ & -\left(k_{\bar{v}} - \beta_{J_f}\left(\lambda_{\bar{d}}\left|\epsilon_{\bar{d}\bar{e}}\right| + \frac{1}{2}\left|\sigma_d\right| + (2 + \frac{\lambda_{\bar{d}}}{2})\beta_{\omega_{l\bar{d}}}\right)\right)\left|\sigma_{\bar{d}}\right|^2 \\
& -\left(l_{\bar{v}}\eta_{\bar{e}l} - k_{\bar{v}} - 2\beta_{J_f}\left(\lambda_{\bar{d}}\left|\epsilon_{\bar{d}\bar{e}}\right| + \beta_{\omega_{l\bar{d}}} + \frac{1}{4}\left|\sigma_d\right|\right)\right)\left|\sigma_{\bar{e}}\right|^2 \\
& -\frac{1}{2}\left(k_{\bar{p}}\eta_{\bar{e}l}\lambda_{\bar{d}} - \lambda_{\bar{d}}\beta_{J_f}\beta_{\omega_{l\bar{d}}}\right)\left|\epsilon_{\bar{d}f}\right|^2 \\
& -\frac{1}{2}\left(l_{\bar{p}}\lambda_{\bar{e}} - \lambda_{\bar{d}}\beta_{J_f}\beta_{\omega_{l\bar{d}}}\right)\left|\epsilon_{\bar{e}f}\right|^2 \\
& -\frac{1}{2}\begin{bmatrix}\left|\epsilon_{\bar{d}f}\right| \\ \left|\epsilon_{\bar{e}f}\right|\end{bmatrix}^\top \begin{bmatrix} k_{\bar{p}}\eta_{\bar{e}l}\lambda_{\bar{d}} & -k_{\bar{p}}\lambda_{\bar{d}} \\ -k_{\bar{p}}\lambda_{\bar{d}} & l_{\bar{p}}\lambda_{\bar{e}}\end{bmatrix}\begin{bmatrix}\left|\epsilon_{\bar{d}f}\right| \\ \left|\epsilon_{\bar{e}f}\right|\end{bmatrix} \\
& +(\left|\sigma_{\bar{d}}\right| + \left|\sigma_{\bar{e}}\right|)\beta_{d_f}.
\end{aligned}
$$

Let $|x_1| \leq \bar{\Delta}_1 \leq 1$. For any $\delta_1 \leq |x_1|$, we define

$$
\begin{aligned}
k_{\bar{v}}^\star :=& \beta_{J_f}\left(\lambda_{\bar{d}} + \frac{1}{2}\bar{\Delta}_1 + (2 + \frac{\lambda_{\bar{d}}}{2})\beta_{\omega_{l\bar{d}}} + \frac{\beta_{d_f}}{\delta_1}\right) \\
l_{\bar{v}}^\star :=& \frac{1}{\sqrt{1-\bar{\Delta}_1^2}}\left(k_{\bar{v}} + 2\beta_{J_f}\left(\lambda_{\bar{d}} + \beta_{\omega_{l\bar{d}}^f} + \frac{1}{4}\bar{\Delta}_1\right) + \frac{\beta_{d_f}}{\delta_1}\right) \\
k_{\bar{p}}^\star :=& \frac{\lambda_{\bar{d}}\beta_{J_f}\beta_{\omega_{l\bar{d}}}}{\lambda_{\bar{d}}\sqrt{1-\bar{\Delta}_1^2}} \\
l_{\bar{p}}^\star :=& \max\left\{\frac{\lambda_{\bar{d}}\beta_{J_f}\beta_{\omega_{l\bar{d}}}}{\lambda_{\bar{e}}}, \frac{k_{\bar{p}}\lambda_{\bar{d}}}{\lambda_{\bar{e}}\sqrt{1-\bar{\Delta}_1^2}}\right\}
\end{aligned}
$$

and choosing the control gains such that $k_{\bar{v}} > k_{\bar{v}}^\star$, $l_{\bar{v}} > l_{\bar{v}}^\star(k_{\bar{v}})$, $k_{\bar{p}} > k_{\bar{p}}^\star$ and $l_{\bar{p}} > l_{\bar{p}}^\star(k_{\bar{p}})$ we satisfy condition (2.23), provided that $x(0) \in \mathcal{B}_{\Delta_1}$ where $\Delta_1 := \sqrt{\overline{\kappa}(\delta)/\underline{\kappa}(\delta)}\bar{\Delta}_1$. Hence, all conditions of Assumption 2.4 are satisfied. Finally, it is shown in Proposition 5.1 that Assumption 2.5 holds, and the conclusion of Proposition 5.2 follows. ■

5.3 Simulation

The spacecraft inertia matrices were chosen to be $J_l = J_f = \text{diag}\{6, 7, 8\}$, where as the input torque were saturated to $\max\{\tau_l\} = \max\{\tau_f\} = 20$. The disturbances acting on the spacecraft, d_l and d_f, were band-limited white noise of power 0.1 and sample time of 0.1 acting about all body frame axis. Examples of disturbances on a spacecraft orbiting Earth are torques

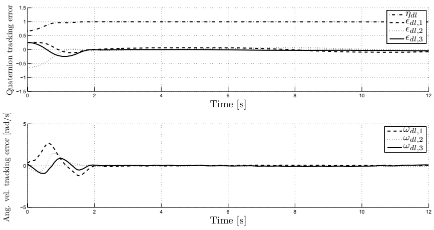

Figure 5.1: Orientation and angular velocity tracking error of the leader spacecraft

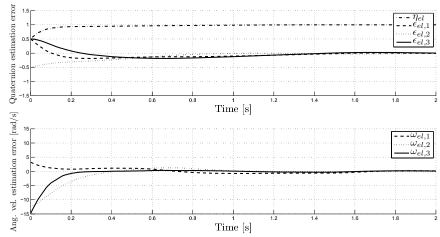

Figure 5.2: Orientation and angular velocity estimation error of the leader spacecraft

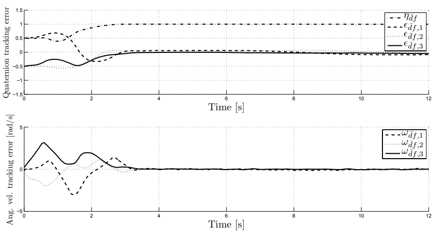

Figure 5.3: Orientation and angular velocity tracking error of the follower spacecraft

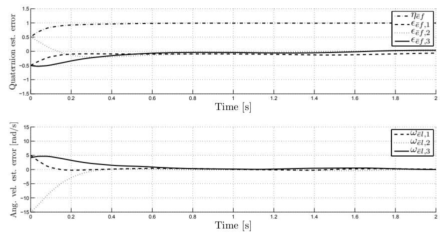

Figure 5.4: Orientation and angular velocity estimation error of the follower spacecraft

due to gravitational, aerodynamic and magnetic forces. The initial conditions for the leader spacecraft model were $q_{il}(0) = \mathrm{col}\,(1/2, 1/2, 1/2, 1/2)$ and $\omega_{il}(0) = \mathrm{col}\,(0.2, 0.3, -0.2)$, where as the controller had initial conditions $q_{ie}(0) = \mathrm{col}\,(1/2, -1/2, 1/2, 1/2)$ and $z(0) = \mathrm{col}\,(5, 6, 4)$ and gains $k_p = 2$, $k_v = 80$, $l_v = 45$, $l_p = 2$, $\lambda_d = 5$ and $\lambda_e = 5$. The reference signal was chosen as $\dot{\omega}^i_{id} = 0.1\,\mathrm{col}(\sin\frac{\pi}{32}t + \frac{\pi}{2}, \sin\frac{\pi}{4}t, \sin\frac{\pi}{8}t + \frac{\pi}{4})$ with ω^i_{id} achieved by numerical integration of $\dot{\omega}^i_{id}$, and the quaternion $q_{id} = \left(\eta_{id}, \epsilon^\top_{id}\right)^\top$ by numerical integration of the relations $\dot{\eta}_{id} = -1/2\epsilon_{id}\omega^i_{id}$ and $\dot{\epsilon}_{id} = 1/2E\left(q_{id}\right)\omega^i_{id}$. Figure 5.1 shows the orientation and angular velocity tracking error of the leader spacecraft. Figure 5.2 shows the estimation errors. The follower spacecraft were chosen to track the orientation and angular velocity of the leader spacecraft. The initial conditions of the follower spacecraft model were $q_{lf}(0) = \mathrm{col}\,(1/2, 1/2, 1/2, -1/2)$ and $\omega_{lf}(0) = \mathrm{col}\,(0, 0, 0)$. The controller initial conditions and gains were $q_{ie}(0) = \mathrm{col}\,(1/2, 1/2, -1/2, -1/2)$ and $z(0) = \mathrm{col}\,(5, 6, 4)$ and $k_{\bar{p}} = k_p$, $k_{\bar{v}} = k_v$, $l_{\bar{v}} = l_v$, $l_{\bar{p}} = l_p$, $\lambda_{\bar{d}} = \lambda_d$ and $\lambda_{\bar{e}} = \lambda_e$, respectively. The gains of the controller and observer were chosen based on the outcome of the

Lyapunov analysis in the previous sections. Figure 5.3 and 5.4 show the simulation results.

Chapter 6

Robustness to a class of signals with unbounded energy

We now exploit the results developed in Section 2.5 to demonstrate the robustness of a spacecraft formation control in a leader-follower configuration, when only position is measured. The focus on output feedback in this illustration is motivated by the fact that velocity measurements in space may not be easily achieved, e.g. can not be equipped with the necessary sensors for such measurements due to space constraints or budget limits. The models described in this section have strong resemblance with the model of a robot manipulator. Our control design is therefore based on control algorithms already validated for robot manipulators, in particular Berghuis and Nijmeijer (1993) and Paden and Panja (1988).

6.1 Disturbances acting on spacecraft formation

From a control design perspective, a crucial challenge is to maintain a predefined relative trajectory, even in presence of disturbances. These disturbances may have diverse origins:

- *Intervehicle interference.* In close formation or spacecraft rendezvous, thruster firings and exhaust gases may influence other spacecraft.
- *Solar wind and radiation.* Particles and radiation expelled from the sun influence the spacecraft and are dependent on the solar activity (Wertz, 1978), which is difficult to predict (Hanslmeier et al., 1999).

- *Small debris.* While large debris would typically mean the end of the mission, some space trash, including paint flakes, dust, coolant and even small needles[1], is small enough to "only" deteriorate the performance, see (NASA, 1999).

- Micrometeoroids. The damages caused to micrometeoroids may be limited due to their tiny size, but constant high velocity impacts will degrade the performance of the spacecraft.

- *Gravitational disturbances.* Even gravitational models including higher order zonal harmonics, can only achieve a certain level of accuracy, as the Earth is neither a sphere nor an ellipsoid, and is certainly not homogeneous, see Figure 6.2. In addition comes the gravitational perturbation due other gravitating bodies such as the Sun and the Moon.

- *Actuator mismatch.* There will commonly be a mismatch between the actuation calculated by the control algorithm, and the actual actuation that the thrusters can provide. This mismatch is particularly present if the control algorithm is based on continuous dynamics, without taking into account pulse based thrusters.

Several of the above mentioned disturbances, have a possibly great amplitude, but last for a short period of time. The class of such signals are well described by the class $\mathcal{W}(E,T)$, described in Section 2.5.

6.2 Model

In this section, we state the models of the spacecraft. The models are similar to the ones derived in (Ploen et al., 2004b). All vectors, both for the leader and the follower spacecraft, are expressed in an orbital frame, with the origin satisfying Newton's gravitational law. The model of the leader is given by

$$m_l\ddot{p} + C_l(\dot{\nu}_o)\dot{p} + D_l(\dot{\nu}_o, \ddot{\nu}_o)p + n_l(r_o, p) = u_l + d_l \tag{6.1}$$

[1]Project West Ford was a test carried out in the early 1960s, where 480 million needles were placed in orbit, with the aim to create an artificial ionosphere above the Earth to allow global radio communication, Overhage and Radford (1964).

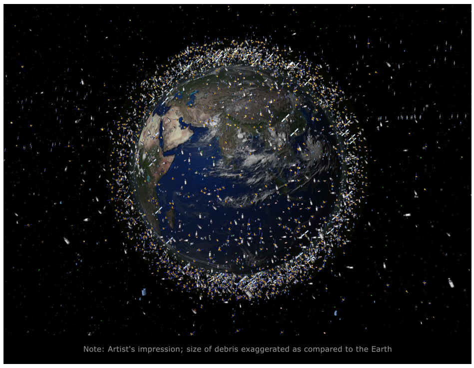

Figure 6.1: Artistic interpretation of debris objects in lo-Earth orbit based on actual density data. Object are however shown at an exaggerated size to make them visible. Reproduced with courtesy to the European Space Agency (http://www.esa.int)

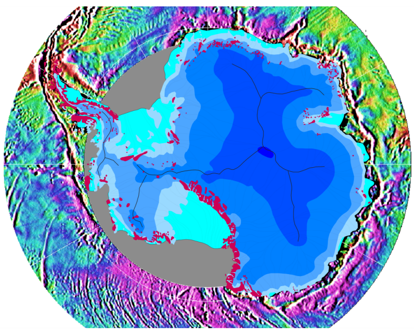

Figure 6.2: Gravity map of the Southern Ocean around the Antarctic continent. This gravity field was computed from sea-surface height measurements collected by the US Navy GEOSAT altimeter between March, 1985, and January, 1990. Reproduced with courtesy to the National Oceanic and Atmospheric Administration (www.noaa.gov)

where ν_o is the true-anomaly of the reference frame,

$$C_l(\dot{\nu}_o) := 2m_l\dot{\nu}_o\bar{C}, \quad \bar{C} := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} D_l(\dot{\nu}_o, \ddot{\nu}_o) \quad &: \quad = m_l\dot{\nu}_o^2\bar{D} + m_l\ddot{\nu}_o\bar{C}, \\ \bar{D} \quad &: \quad = \operatorname{diag}(-1, -1, 0) \in \mathbb{R}^{3\times 3}, \end{aligned}$$

and

$$n_l(r_o, p) := m_l\mu\left(\frac{r_o + p}{|r_o + p|^3} - \frac{r_o}{|r_o|^3}\right).$$

The model describing the motion of the follower spacecraft relative to the leader is given by:

$$m_f\ddot{\rho} + C_f(\dot{\nu}_o)\dot{\rho} + D_f(\dot{\nu}_o, \ddot{\nu}_o)\rho + n_f(r_o, p, \rho) = u_f + d_f - \frac{m_f}{m_l}(u_l + d_l), \quad (6.2)$$

with

$$D_f(\dot{\nu}_o, \ddot{\nu}_o) := m_f\dot{\nu}_o^2\bar{D} + m_f\ddot{\nu}_o\bar{C}$$

and

$$n_f(r_o, p, \rho) := m_f\mu\left(\frac{r_o + p + \rho}{|r_o + \rho + p|^3} - \frac{r_o + p}{|r_o + p|^3}\right).$$

In the analysis of the formation we will make use of the following assumption:

Assumption 6.1 *The true anomaly rate $\dot{\nu}_o$, and true anomaly rate-of-change, $\ddot{\nu}_o$, of the reference frame, are upper bounded by a constant, that is $|\dot{\nu}_o(t)| \leq \beta_{\dot{\nu}_o}$ and $|\ddot{\nu}_o(t)| \leq \beta_{\ddot{\nu}_o}$ for all $t \geq t_0 \geq 0$, for some positive constants $\beta_{\dot{\nu}_o}$ and $\beta_{\ddot{\nu}_o}$.*

Note that this assumption is naturally satisfied when the reference frame is following a Keplerian orbit, but it also holds for any sufficiently smooth reference trajectory.

6.3 Control of the leader spacecraft

We now propose a controller whose goal is to make the leader spacecraft follow a trajectory given by the desired position $p_d(t)$ relative to the orbit frame. Let $\hat{p} \in \mathbb{R}^3$ be the estimated position. We define $e_l := p - p_d$ as the tracking error and $\tilde{p} := p - \hat{p}$ as the estimation error. Similarly to Berghuis and Nijmeijer (1993), the controller is given by:

$$u_l = m_l \ddot{p}_d + C_l(\dot{\nu}_o)\dot{p}_d + D_l(\dot{\nu}_o, \ddot{\nu}_o)p + n_l(r_o, p) - K_l(\dot{p}_0 - \dot{p}_r) \tag{6.3}$$
$$\dot{p}_r = \dot{p}_d - \ell_l e_l \tag{6.4}$$
$$\dot{p}_0 = \dot{\hat{p}} - \ell_l \tilde{p}, \tag{6.5}$$

with $K_l := k_l I_{3\times3}$ and $k_l, \ell_l > 0$. The observer is given by:

$$\dot{\hat{p}} = a_l + (l_l + \ell_l)\tilde{p} \tag{6.6}$$
$$\dot{a}_l = \ddot{p}_d + l_l \ell_l \tilde{p} \tag{6.7}$$

where $l_l > 2k_l/m_l$ denotes a free positive constant. Let us define $x_2 := \left(\dot{e}_l^\top, \ell_l e_l^\top, \dot{\tilde{p}}^\top, \ell_l \tilde{p}^\top\right)^\top \in \mathbb{R}^{12}$. We can now write the system in a state space form $\dot{x}_2 = f_2(x_2, d_2)$ where[2]

$$f_2(t, x_2, d_2) := \begin{bmatrix} \frac{1}{m_l}\sigma_1(t, x_2, d_2) \\ \ell_l \dot{e}_l \\ \frac{1}{m_l}\sigma_3(t, x_2, d_2) \\ \ell_l \dot{\tilde{p}} \end{bmatrix} \tag{6.8}$$

with $d_2 = d_l$ and

$$\begin{aligned} \sigma_1(t, x_2, d_2) :=& \quad -C_l \dot{e}_l - K_l Y x_2 + d_2 \\ \sigma_3(t, x_2, d_2) :=& \quad -C_l \dot{e}_l - m_l \ell_l \dot{\tilde{p}} - K_l Y x_2 - m_l l_l \left(\ell_l \tilde{p} + \dot{\tilde{p}}\right) + d_2, \\ Y :=& \quad \begin{bmatrix} I_{3\times3} & -I_{3\times3} & I_{3\times3} & -I_{3\times3} \end{bmatrix}. \end{aligned}$$

6.4 Control of the follower spacecraft

We now propose a controller to make the follower spacecraft track the trajectory given by the desired position $\rho_d(t)$ relative to the leader. Let $\hat{\rho} \in \mathbb{R}^3$ be the estimated position. We define $e_f := \rho - \rho_d$ as the tracking error and $\tilde{\rho} := \rho - \hat{\rho}$ as the estimation error. We use a similar controller as for the leader spacecraft, that is:

[2] The results of Section 2.5 are stated in a time-invariant setup, but can naturally be extended to non-autonomous systems if the properties on the considered Lyapunov function hold uniformly in time which, as we will see, in the case here.

$$
\begin{aligned}
u_f =& m_f \ddot{\rho}_d + C_f\left(\dot{\nu}_o\right)\dot{\rho}_d + D_f\left(\dot{\nu}_o, \ddot{\nu}_o\right)\rho + n_f\left(r_o, p, \rho\right) + \frac{m_f}{m_l} n_l\left(r_o, p\right) \\
& - K_f\left(\dot{\rho}_0 - \dot{\rho}_r\right) && (6.9) \\
\dot{\rho}_r =& \dot{\rho}_d - \ell_f e_f && (6.10) \\
\dot{\rho}_0 =& \dot{\hat{\rho}} - \ell_f \tilde{\rho}, && (6.11)
\end{aligned}
$$

where $K_f := k_f I_{3\times 3}$, $k_f, \ell_f > 0$ being tuning gains. The observer is given by

$$
\begin{aligned}
\dot{\hat{\rho}} &= a_f + \left(l_f + \ell_f\right)\tilde{\rho}_f && (6.12) \\
\dot{a}_f &= \ddot{\rho}_d + l_f \ell_f \tilde{\rho}_f && (6.13)
\end{aligned}
$$

with a positive constant $l_f > 2k_f/m_f$. Define $x_1 := \left(\dot{e}_f^\top, \ell_f e_f^\top, \dot{\tilde{\rho}}^\top, \ell_f \tilde{\rho}^\top\right)^\top \in \mathbb{R}^{12}$. Combining (6.2) and (6.9-6.13) and inserting the leader spacecraft controller u_l by (6.3), we can summarize the follower spacecraft's dynamics by the state space equation $\dot{x}_1 = f_1\left(x_1, d_1\right)$ where

$$
f_1\left(t, x_1, x_2, d_1\right) := \begin{bmatrix} \frac{1}{m_f}\sigma_1(t, x_1, x_2, d_1) \\ \ell_f \dot{e}_f \\ \frac{1}{m_f}\sigma_3(t, x_1, x_2, d_1) \\ \ell_f \dot{\tilde{p}} \end{bmatrix}, \tag{6.14}
$$

with

$$
\begin{aligned}
\sigma_1(t, x_1, x_2, d_1) :=& \quad -C_f \dot{e}_f - K_f Y x_1 \\
& \quad -m_f(\dot{\nu}_o(t)^2 \bar{D} + \ddot{v}_o(t)\bar{C})e_l + K_l Y x_2 + d_1 \\
\sigma_3(t, x_1, x_2, d_1) :=& \quad -C_f \dot{e}_f - m_f \ell_f \dot{\tilde{\rho}} - K_f Y x_1 - m_f l_f \left(\ell_f \tilde{\rho} + \dot{\tilde{\rho}}\right) \\
& \quad -m_f(\dot{\nu}_o(t)^2 \bar{D} + \ddot{v}_o(t)\bar{C})e_l + K_l Y x_2 + d_1 \\
d_1 :=& \quad -m_f \ddot{p}_d - m_f(\dot{\nu}_o^2 \bar{D} + \ddot{v}_o \bar{C})p_d - \frac{m_f}{m_l} d_l + d_f \,. && (6.15)
\end{aligned}
$$

6.5 Stability analysis of the overall formation

We are now ready to state the following result, which establishes the robustness of the controlled formation to a wide class of disturbances.

Proposition 6.1 *Let Assumption 6.1 hold. Let the controller of the leader spacecraft be given by (6.3)-(6.7) with* $l_l \geq 2k_l/m_l$ *and* $k_l > 9/2k_l^\star$, *where*

$$k_l^\star := m_l\ell_l + 4m_l\beta_{\dot{\nu}_o} + 2\frac{m_f}{\ell_l}\left(\beta_{\dot{\nu}_o}^2 + \beta_{\ddot{\nu}_o}\right) + 2 \tag{6.16}$$

and let the controller of the follower spacecraft be given by (6.9)-(6.13) with $l_f \geq 2k_f/m_f$ *and* $k_f > 3k_f^\star/2$, *where*

$$k_f^\star := m_f\ell_f + 4m_f\beta_{\dot{\nu}_o} + 2\frac{m_f}{\ell_l}\left(\beta_{\dot{\nu}_o}^2 + \beta_{\ddot{\nu}_o}\right) + 2 + \frac{3}{2}k_l. \tag{6.17}$$

Given any precision $\delta > 0$ *and any time window* $T > 0$, *consider any average excitation satisfying*

$$E(T,\delta) \leq \frac{\min\{m_f, m_l\}\,\delta^2}{12}\frac{e^{\kappa T}-1}{2e^{\kappa T}-1}, \tag{6.18}$$

where

$$\kappa := \frac{1}{6}\frac{k_l^\star}{\max\{k_l/\ell_l, k_f/\ell_f\}}. \tag{6.19}$$

Then, for any $d \in \mathcal{W}_\gamma(E,T)$ *where* $\gamma(s) := 2s^2$, *the ball* $\overline{\mathcal{B}}_\delta$ *is GAS for the overall formation* $\dot{x} = f(t,x,d)$ *with* $x := (x_1^\top, x_2^\top)^\top$, $d := \left(d_1^\top, d_2^\top\right)^\top$ *and* $f := (f_1^\top, f_2^\top)$.

Proof. The proof is done by applying Corollary 2.1. We consider the Lyapunov function candidate

$$V(x) := \frac{1}{2}V_1(x_1) + \frac{1}{2}V_2(x_2) \tag{6.20}$$

where $V_1(x_1) := x_1^\top W^\top R_1 W x_1$ and $V_2(x_2) := x_2^\top W^\top R_2 W x_2$, with $R_1 := \mathrm{diag}(m_f I_{3\times3}, 2K_f/\ell_f - m_f I_{3\times3}, m_f I_{3\times3}, 2K_f/\ell_f)$, $R_2 := \mathrm{diag}(m_l I_{3\times3}, 2K_l/\ell_l - m_l I_{3\times3}, m_l I_{3\times3}, 2K_l/\ell_l)$ and

$$W := \begin{bmatrix} I_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_{3\times3} & I_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_{3\times3} \end{bmatrix}.$$

In order to simplify the Lyapunov analysis, define the sliding variables $t_1 := \dot{\rho} - \dot{\rho}_r$ and $t_2 := \dot{\rho} - \dot{\rho}_0$ in such a way that $t_1 = \dot{e}_f + \ell_f e_f$ and $t_2 = \dot{\tilde{\rho}} + \ell_f\tilde{\rho}$. Define also $s_1 := \dot{p} - \dot{p}_r$ and $s_2 := \dot{p} - \dot{p}_0$ such that $s_1 = \dot{e}_l + \ell_l e_l$

and $s_2 = \dot{\tilde{p}} + \ell_l \tilde{p}$. It can be shown that the time derivative of the Lyapunov function candidate can be compactly written as

$$\begin{aligned}
\dot{V} &= -x_1^\top Q_f x_1 - t_2^\top \left(l_f m_f I_{3\times 3} - 2K_f\right) t_2 \\
&\quad + (t_1 + t_2)^\top \left(d_1 - C_f\left(\dot{\nu}_o\right)\dot{e}_f\right) \\
&\quad + (t_1 + t_2)^\top \left(-m_f(\dot{\nu}_o^2 \bar{D} + \ddot{v}_o \bar{C})e_l + K_l Y x_2\right) \\
&\quad - x_2^\top Q_l x_2 - s_2^\top \left(l_l m_l I_{3\times 3} - 2K_l\right) s_2 \\
&\quad - (s_1 + s_2)^\top \left(d_2 - C_l\left(\dot{\nu}_o\right)\dot{e}_l\right)
\end{aligned}$$

with $Q_l := \mathrm{diag}(K_l - m_l \ell_l I_{3\times 3}, K_l, K_l, K_l) \in \mathbb{R}^{12\times 12}$ and $Q_f := \mathrm{diag}(K_f - m_f \ell_f I_{3\times 3}, K_f, K_f, K_f) \in \mathbb{R}^{12\times 12}$. Note that $|t_1 + t_2| \leq 2|x_1|$, $|s_1 + s_2| \leq 2|x_2|$, $|\dot{e}_f| \leq |x_1|$, $|\dot{e}_l| \leq |x_2|$, and that $|e_l| \leq |x_1|/\ell_l$. Recalling that $l_f \geq 2k_f/m_f$ and $l_l \geq 2k_l/m_l$, and invoking Assumption 6.1, we get that the derivative of the Lyapunov function can be upper bounded as:

$$\begin{aligned}
\dot{V} &\leq -\left(k_f - m_f \ell_f - 4m_f \beta_{\dot{\nu}_o}\right)|x_1|^2 \\
&\quad +2|d_1||x_1| + 2\frac{m_f}{\ell_l}\left(\beta_{\dot{\nu}_o}^2 + \beta_{\ddot{\nu}_o}\right)|x_1||x_2| + 2k_l|x_1||x_2| \\
&\quad -\left(k_l - m_l \ell_l - 4m_l \beta_{\dot{\nu}_o}\right)|x_2|^2 \\
&\quad +2|d_2||x_2|.
\end{aligned} \tag{6.21}$$

By Young's inequality, $|d_1||x_1| \leq |d_1|^2 + |x_1|^2$, $|d_2||x_2| \leq |d_2|^2 + |x_2|^2$, $|x_1||x_2| \leq |x_1|^2 + |x_2|^2$, and noticing that $k_l|x_1||x_2| \leq k_l/3|x_2|^2 + 3k_l/4|x_1|^2$, it follows that

$$\begin{aligned}
\dot{V} &\leq -\left(k_f - m_f \ell_f - 4m_f \beta_{\dot{\nu}_o} - 2\frac{m_f}{\ell_l}\left(\beta_{\dot{\nu}_o}^2 + \beta_{\ddot{\nu}_o}\right) - 2 - \frac{3}{2}k_l\right)|x_1|^2 \\
&\quad -\left(k_l\left(1 - \frac{2}{3}\right) - m_l \ell_l - 4m_l \beta_{\dot{\nu}_o} - 2\frac{m_f}{\ell_l}\left(\beta_{\dot{\nu}_o}^2 + \beta_{\ddot{\nu}_o}\right) - 2\right)|x_2|^2 \\
&\quad +2|d_1|^2 + 2|d_2|^2 .
\end{aligned}$$

If we chose $k_l > 9/2k_l^\star$ and $k_f > 3k_f^\star/2$ as given in the statement of Proposition 6.1, we are ensured that R_1, R_2, Q_l, Q_f are all positive definite matrices, and it can be seen that

$$\frac{1}{6}\min\{m_l, m_f\}|x|^2 \leq V(x) \leq 3\max\{k_l/\ell_l, k_f/\ell_f\}|x|^2 .$$

Using these inequalities and the fact that $k_l^\star < k_f^\star$, we get that

$$\begin{aligned} \dot{V} &\leq -\frac{1}{2}k_l^\star \left(|x_1|^2 + |x_2|^2\right) + 2|d_1|^2 + 2|d_2|^2 \\ &\leq -\frac{1}{2}k_l^\star |x|^2 + 2|d|^2 \\ &\leq -\kappa V(x) + 2|d|^2 \end{aligned} \tag{6.22}$$

with the constant κ defined in (6.19). Hence, the conditions of Corollary 2.1 are satisfied, with $\underline{\alpha}(s) = \frac{1}{6}\min\{m_l, m_f\}s^2$, $\overline{\alpha}(s) = 3\max\{k_l/\ell_l, k_f/\ell_f\}s^2$ and $\gamma(s) = 2s^2$, and the conclusion follows. ■

6.6 Simulations

Let the reference orbit be an eccentric orbit with radius of perigee $R_p = 10^7 m$ and radius of apogee $R_a = 3 \times 10^7 m$, which can be generated by numerical integration of

$$\ddot{r}_o = -\frac{\mu}{|r_o|^3} r_o, \tag{6.23}$$

with $r_o(0) = (R_p, 0, 0)$ and $\dot{r}_o(0) = (0, V_p, 0)$, and where

$$V_p = \sqrt{2\mu\left(\frac{1}{R_p} - \frac{1}{(R_p + R_a)}\right)}.$$

For simplicity, we choose the desired trajectory of the leader spacecraft to coincide with the reference orbit. The initial values of the leader spacecraft are $p_l(0) = (2, -2, 3)^\top$ and $\dot{p}_l(0) = (0.4, -0.8, -0.2)^\top$. The initial values of the observer are chosen as $\hat{p}(0) = (0, 0, 0)^\top$ and $a_l(0) = (0, 0, 0)^\top$.

The reference trajectory of the follower spacecraft are chosen as the solutions of a special case of the Clohessy-Wiltshire equations, cf. Clohessy and Wiltshire (1960). We use

$$\rho_d(t) = \begin{bmatrix} 10\cos\nu_o(t) \\ -20\sin\nu_o(t) \\ 0 \end{bmatrix}. \tag{6.24}$$

Here, ν_o is the true anomaly of the reference frame, obtained by numerical integration of the equation

$$\ddot{\nu}_o(t) = \frac{-2\mu\left(1 + e_o\cos\nu_o(t)\right)^3 \sin\nu_o(t)}{(R_p + R_a)^3 (1 - e_o^2)^3}.$$

Since the reference frame is initially at perigee, $\nu_o(0) = 0$ and $\dot{\nu}_o(0) = V_p/R_p$. The eccentricity of the reference frame is constant, and can be calculated from R_a and R_p to be $e_o = 0.5$. This choice of reference orbit means that the two spacecraft are in the same orbital plane, and that the follower spacecraft will make a full rotation about the leader spacecraft per orbit around the Earth. The initial values of the follower spacecraft are $\rho(0) = (9, -1, 2)^\top$ and $\dot{\rho}(0) = (-0.3, 0.2, 0.6)^\top$. The initial parameters of the observer are chosen to be $\hat{\rho}(0) = \rho_d(0) = (10, 0, 0)^\top$ and $a_f(0) = (0, 0, 0)^\top$. We use $m_f = m_l = 25$ kg both in the model and the control structure. The control gains are based on the analysis in Section 6.5. First we pick $\ell_l = 0.06$. Then, by assuming we can ignore the effect of $\dot{\nu}_o$ and $\ddot{\nu}_o$ (due to the great orbit which implies that $\dot{\nu}_o$ and $\ddot{\nu}_o$ vary slowly), we find that $k_l^\star = 3.5$ from (6.16). Since k_l should satisfy $k_l > 9/2k_l^\star$, we chose $k_l = 15.75$. We now pick $\ell_f = 0.15$, and again by assuming that we can ignore the effect of $\dot{\nu}_o$ and $\ddot{\nu}_o$, we find from (6.16) that $k_f^\star = 29.375$. Since k_f should satisfy $k_f > 3/2k_f^\star$, we chose $k_f = 44.1$. At last we take $l_f = 3.52$ and $l_l = 1.26$. With these choices, we find from (6.19) that $\kappa = 0.0019841$. Over a 10 second interval the average excitation must satisfy $E(T, \delta) \leq 0.04014\delta^2$, according to (6.18). We consider two types of disturbances acting on the spacecraft; impacts and continuous disturbances. The impacts have random amplitude, but with maximum of 1.5 N in each direction of the Cartesian frame. Out of simplicity we allow only one impact over each 10 second interval, and we assume that the duration of each impact is 0.1 s. The continuous part is taken as sinusoids, also acting in each direction of the Cartesian frame, and are chosen to be $(0.1 \sin 0.01t, 0.25 \sin 0.03t, 0.3 \sin 0.04t)^\top$ for both spacecraft. The motivation for choosing the same kind of continuous disturbance for both spacecraft, is that this disturbance is typically due to gravitational perturbation, which at least for close formations, have the same effect on both spacecraft. Notice from (6.2) that the relative dynamics are influenced by disturbances acting on the leader- and follower spacecraft, so the effect of the continuous part of the disturbance on the relative dynamics is zero.

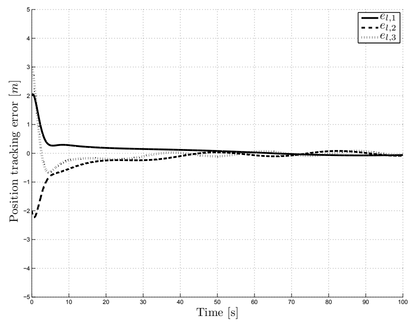

Figure 6.3: Position tracking error of leader spacecraft

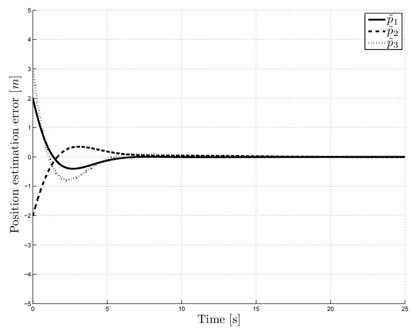

Figure 6.4: Position estimation error of follower spacecraft

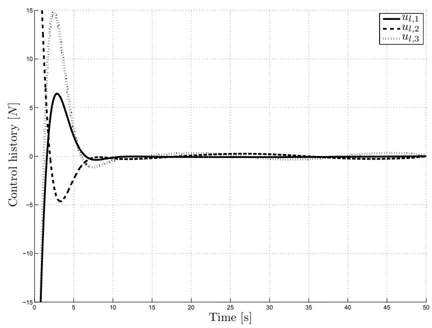

Figure 6.5: Control forces acting on leader spacecraft

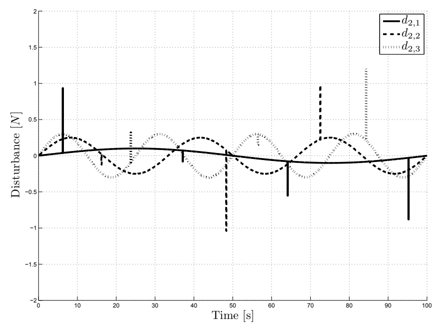

Figure 6.6: Disturbance forces acting on leader spacecraft

The disturbances satisfy the following average excitation

$$\begin{aligned}\int_t^{t+10} 2|d(\tau)|^2 \mathrm{d}\tau \leq & 2\int_t^{t+10} (0.1 \sin 0.01\tau)^2 \mathrm{d}\tau \\ & + 2\int_t^{t+10} (0.25 \sin 0.03\tau)^2 \mathrm{d}\tau \\ & + 2\int_t^{t+10} (0.3 \sin 0.04\tau)^2 \mathrm{d}\tau \\ & + 3\int_t^{t+10} 2(1.5^2) \mathrm{d}\tau \\ \leq & 1.48.\end{aligned}$$

This means that we can expect a precision better than $\delta = 6.1$.

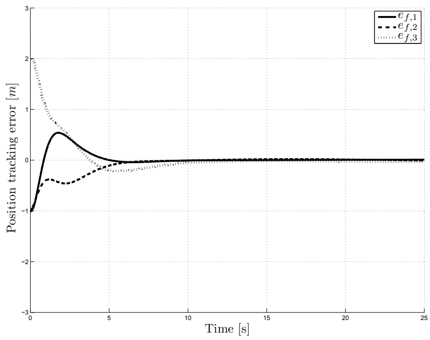

Figure 6.7: Position tracking error of follower spacecraft

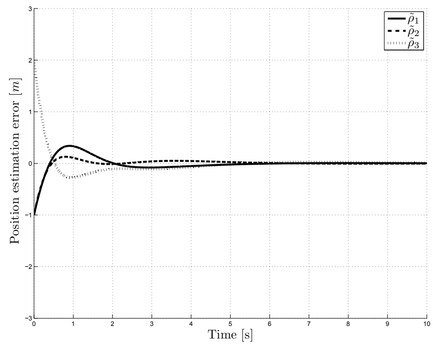

Figure 6.8: Position estimation error of follower spacecraft

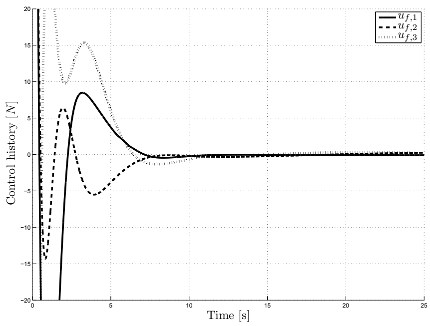

Figure 6.9: Control forces acting on follower spacecraft

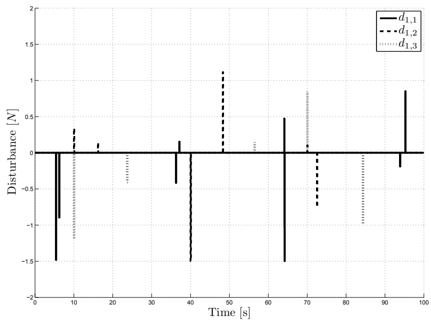

Figure 6.10: Disturbance forces acting on follower spacecraft

Figure 6.3, 6.4 and 6.5 show the position tracking error, position estimation error and control history of the leader spacecraft, where as Figure 6.7, 6.8 and 6.9 are the equivalent figures for the follower spacecraft. Figure 6.6 and 6.10 show the effect of d_2 and d_1 acting on the formation. Notice in Figure 6.10 that the effect of the continuous part of the disturbance is canceled out (since we consider relative dynamics and both spacecraft are influenced by the same continuous disturbance), where as the effect of the impacts has increased compared to the effect of the impacts on the leader spacecraft. The control gains have been chosen based on the Lyapunov analysis. This yields in general very conservative constraints on the choice of control gains, and also conservative estimates of the disturbances the control system is able to handle. As shown in Figure 6.5, and in particular Figure 6.9, this leads to large transients in the actuation.

Chapter 7

Quantized and pulse width modulated actuation

As propulsion systems of spacecraft often do not provide continuous actuation, we will in this chapter show how strong stability properties can be achieved also under certain quantization schemes or pulse width modulation of the continuous control signal.

7.1 Solutions of discontinuous differential equations

Due to the discontinuous feedback considered in this chapter, we will have to recapture some mathematical tools from set-valued analysis. Stability of sets (both bounded and unbounded) have been treated extensively in the literature, see for instance Lin (1992), Lin et al. (1996), Vorotnikov (1998), Teel and Praly (2000), Teel et al. (2002), Chellaboina and Haddad (2002), Skjetne (2005) and Tjønnås (2008) for developments in the last two decades. Let $F : \mathcal{G} \rightarrow$(subsets of $\mathbb{R}^n$), where $\mathcal{G}$ is an open subset of $\mathbb{R}^n$. We will make use of the following definitions (Teel and Praly (2000),Teel et al. (2004)).

Definition 7.1 *The set-valued map F is said to be* upper semicontinuous *on $\mathcal{G}$ if, given $x \in \mathcal{G}$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $F\left(\mathcal{B}_\delta\left(x\right)\right) \subset F\left(x\right) + \mathcal{B}_\varepsilon$.*

Definition 7.2 *The set-valued map F is said to* satisfy the basic conditions *on $\mathcal{G}$ if it is upper semicontinuous on $\mathcal{G}$ and, for each $x \in \mathcal{G}$, $F\left(x\right)$ is nonempty, compact and convex.*

Given a nonlinear discontinuous differential equation

$$\dot{x} = f(t, x) \tag{7.1}$$

where $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 7.3 *An absolutely continuous function $x(t, x_0, t_0)$ is said to be a solution of (7.1)* in the sense of Krasowskii*, if for almost every*[1] $t \geq t_0 \geq 0$ *satisfies the differential inclusion*

$$\dot{x} \in \mathfrak{K}(f(t, x)) := \bigcap_{\delta > 0} \overline{\text{co}} f(t, \mathcal{B}_\delta(x))$$

Definition 7.4 *An absolutely continuous function $x(t, x_0, t_0)$ is said to be a solution of (7.1)* in the sense of Filippov*, if for almost every $t \geq t_0 \geq 0$ satisfies the differential inclusion*

$$\dot{x} \in \mathfrak{F}(f(t, x)) := \bigcap_{\delta > 0} \bigcap_{\text{meas}(N)=0} \overline{\text{co}} f(t, \mathcal{B}_\delta(x) \setminus N).$$

The intersection is taken over all sets N of measure 0 to be able to exclude sets on which $f(t, x)$ is not defined, see Sastry (1999). Local existence of Krasowskii and Filippov solutions are ensured by the following theorem:

Theorem 7.1 (Ceragioli, 1999, Theorem 5) *If $f : [t_0, t_0 + a] \times \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded (locally essentially bounded), then a local Krasovskii (Filippov) solution of (7.1) exists.*

For more on existence (or uniqueness) of solutions to differential inclusions the reader is referred to Cortés (2008), Ceragioli (1999) and Filippov (1988).

7.2 Quantized actuation

An interesting approach for the analysis of systems with discontinuous actuation are found in Ceragioli and De Persis (2007), where sufficient conditions for a stabilizing feedback law to be *quantizable* can be found. The rest of this section will build upon the results of this paper.

[1] "almost every" means except for a set of t of measure 0.

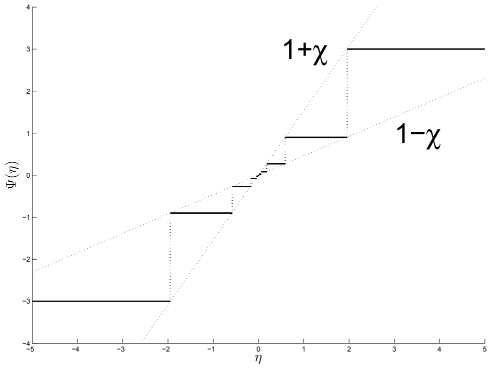

Figure 7.1: Quantization levels and sector bounds

We consider a special case of (7.1), namely the parameterized input affine system

$$\begin{aligned}\dot{x} &= f(t,x,\theta) \\ &= \bar{f}(t,x) + g(x)\,u(t,x,\theta) \end{aligned} \tag{7.2}$$

(with $\bar{f} \in \mathbb{R}^n \to \mathbb{R}^n$ and $g \in \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuously differentiable, and $u \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathcal{U} \subset \mathbb{R}^m$ with $\mathcal{U}$ is the set of all measurable and locally bounded functions) or, in case of quantized feedback:

$$\begin{aligned}\dot{x} &= f(t,x,\theta) \\ &= \bar{f}(t,x) + g(x)\,\Psi(u(t,x,\theta)) \end{aligned} \tag{7.3}$$

where the quantization scheme $\Psi(\cdot)$ in this paper is as follows Ceragioli and De Persis (2007): Let the input signal to be quantized be η. Let $\chi = (1-\rho)/(1+\rho)$ with $0 < \rho < 1$ fixed, and let $u_i = \rho^i u_0$ with $i \in \mathbb{Z}$ and $u_0 > 0$. Then the output of the quantizer is:

$$\Psi(\eta) = \begin{cases} u_i & \text{if} \quad \frac{1}{1+\chi}u_i \leq \eta \leq \frac{1}{1-\chi}u_i \\ 0 & \text{if} \quad \eta = 0 \\ -\Psi(-\eta) & \text{if} \quad \eta \leq 0 \end{cases} \tag{7.4}$$

which can be seen in Figure 7.1. For a vectorial input signal the quantizer is applied on each element of the vector. The Krasowskii solutions of (7.3) are absolutely continuous functions which satisfies the differential inclusion:

$$\begin{aligned}\dot{x} &\in \bar{f} + gK(\Psi(u)) \\ &\subseteq \bar{f} + (1+\lambda\chi)\,gu \end{aligned} \tag{7.5}$$

with $\lambda \in [-1, 1]$, and where the dependencies of $\bar{f}$, g and u have been left out for notational simplicity.

We can now state the following proposition, which gives sufficient conditions for a system to be UGPES on a parameter set Θ, with respect to Krasowskii solutions.

Proposition 7.1 *Let Θ be a subset of $\mathbb{R}^p$ and suppose that, given any $\delta > 0$, there exist a parameter $\theta^\star(\delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and positive constants $k_1(\delta), k_2(\delta), k_3(\delta), k_4(\delta)$ such that for all $x \in \mathbb{R}^n \backslash \mathcal{B}_\delta$ and all $t \in \mathbb{R}_{\geq 0}$,*

$$k_1(\delta)|x|^p \leq V_\delta(t, x) \leq k_2(\delta)|x|^p \tag{7.6}$$

$$\frac{\partial V_\delta}{\partial t}(t, x) + \frac{\partial V_\delta}{\partial x}(t, x)\left(f(t, x, \theta^\star)\right) \leq -k_3(\delta)|x|^p \tag{7.7}$$

with $f(t, x, \theta^\star) = \bar{f}(t, x) + g(x)\, u(t, x, \theta^\star)$,

$$\frac{\partial V_\delta}{\partial x}(t, x)\, g(x)\, u(t, x, \theta^\star) \leq k_4(\delta)|x|^p \tag{7.8}$$

where p denotes a positive constant. Then, under the condition that

$$\lim_{\delta \to 0} \frac{k_2(\delta)\, \delta^p}{k_1(\delta)} = 0 \tag{7.9}$$

the system $\dot{x} = \bar{f}(t, x) + g(x)\, \Psi(u(t, x, \theta))$ is UGPES on the parameter set Θ with respect to Krasowskii solutions.

Proof. Let $v \in K(\Psi(u(t, x, \theta)))$. Consider the case where $\frac{\partial V_\delta}{\partial x} g(x)\, u(t, x, \theta) \neq 0$. Due to (7.5) we have that (leaving out the arguments of brevity)

$$\frac{\partial V_\delta}{\partial x}\left(\bar{f} + gv\right) = \frac{\partial V_\delta}{\partial x}\left(\bar{f} + gu\right) + \lambda\chi\frac{\partial V_\delta}{\partial x} gu$$

Using (7.7), (7.8) and that $|\lambda| \leq 1$ we get that

$$\begin{aligned} \frac{\partial V_\delta}{\partial t} + \frac{\partial V_\delta}{\partial x}\left(\bar{f} + gv\right) &\leq -k_3(\delta)|x|^p + |\lambda|\, \chi k_4(\delta)|x|^p \\ &\leq -\left(k_3(\delta) - \chi k_4(\delta)\right)|x|^p \end{aligned}$$

We see that the conditions of Theorem 2.1 in Chapter 2 are satisfied with $\underline{\kappa}(\delta) = k_1(\delta), \overline{\kappa}(\delta) = k_2(\delta), \kappa(\delta) = k_3(\delta) - \chi k_4(\delta)$. Notice that $\kappa(\delta)$ can always be chosen positive, by increasing ρ of the quantizer and thereby decreasing χ. When $\frac{\partial V_\delta}{\partial x} gu = 0$, then from (7.7) we have that

$$\begin{aligned}\frac{\partial V_\delta}{\partial t} + \frac{\partial V_\delta}{\partial x}\left(\bar{f} + gv\right) &= \frac{\partial V_\delta}{\partial t} + \frac{\partial V_\delta}{\partial x}\bar{f} \\ &\leq -k_3\left(\delta\right)\left|x\right|^p\end{aligned}$$

so the conditions of Theorem 2.1 are again satisfied, this time with $\kappa(\delta) = k_3(\delta)$, and the conclusion follows. ■

Remark 7.1 *The proof of Proposition 7.1 follows along the same lines as the proof of (Ceragioli and De Persis, 2007, Proposition 1) for global asymptotic stability. As mentioned in (Ceragioli and De Persis, 2007, Remark below Proposition 1), the result is easily extended to exponential stability provided that the system (7.2) is exponentially stable.*

7.2.1 Model of follower spacecraft

We only consider the relative dynamics of the two spacecraft, which is given by

$$m_f\ddot{\rho} + C_f\left(\dot{\nu}_l\right)\dot{\rho} + D_f\left(\dot{\nu}_l, \ddot{\nu}_l, r_l, \rho\right)\rho + n_f\left(r_l, \rho\right) = u_f + d_f - \frac{m_f}{m_l}\left(u_l + d_l\right),$$

with ρ being the relative position between the spacecraft.

7.2.2 Control scheme

We now show how similar results as achieved in Proposition 4.2 for continuous actuation, under certain additional assumptions also hold for quantized actuation. To keep this section self contained, we restate the assumptions of Proposition 4.2 here.

Assumption 7.1 *Define $\tilde{\nu}\left(t\right) := \nu_l - \nu_d$, where $\nu_l\left(t\right)$ and $\nu_d\left(t\right)$ are the actual and the desired true anomaly of the leader spacecraft, respectively. We will assume that the desired true anomaly rate and true anomaly rate-of-change of the leader spacecraft is bounded, i.e. given some positive constant r_1, r_2, $\left|\dot{\nu}_d\left(t_0\right)\right| \leq r_1$ and $\left|\ddot{\nu}_d\left(t_0\right)\right| \leq r_2$ implies that $\left|\dot{\nu}_d\left(t\right)\right| \leq \beta_{\dot{\nu}_d}$ and $\left|\ddot{\nu}_d\left(t\right)\right| \leq \beta_{\ddot{\nu}_d}$ for all $t \geq t_0 \geq 0$, for some positive constants $\beta_{\dot{\nu}_d}$ and $\beta_{\ddot{\nu}_d}$. Furthermore, we assume that the actuation system of the leader spacecraft keeps $\dot{\tilde{\nu}}$ and $\ddot{\tilde{\nu}}$ bounded, i.e. given some positive constant r_3, r_4, $\left|\dot{\tilde{\nu}}\left(t_0\right)\right| \leq r_3$ implies that $\left|\dot{\tilde{\nu}}\left(t\right)\right| \leq \beta_{\dot{\tilde{\nu}}}$ for all $t \geq t_0 \geq 0$, and $\left|\ddot{\tilde{\nu}}\left(t_0\right)\right| \leq r_4$ implies that $\left|\ddot{\tilde{\nu}}\left(t\right)\right| \leq \beta_{\ddot{\tilde{\nu}}}$ for all $t \geq t_0 \geq 0$, where $\beta_{\dot{\tilde{\nu}}}, \beta_{\ddot{\tilde{\nu}}}$ are positive constants.*

Again we emphasize that this is no restriction on the initial relative state vector, to be defined in the sequel. In addition we will make the following assumption regarding the desired trajectories of the follower spacecraft:

Assumption 7.2 *The desired relative position $\rho_d(t)$, desired relative velocity $\dot{\rho}_d(t)$ and desired relative acceleration $\ddot{\rho}_d(t)$ are all smooth and bounded functions, i.e. there exists positive constants β_{ρ_d}, $\beta_{\dot{\rho}_d}$, $\beta_{\ddot{\rho}_d}$ such that $|\rho_d(t)| \leq \beta_{\rho_d}$, $|\dot{\rho}_d(t)| \leq \beta_{\dot{\rho}_d}$ and $|\ddot{\rho}_d(t)| \leq \beta_{\ddot{\rho}_d}$ for all $t \geq t_0 \geq 0$.*

We will also assume that the disturbances acting on the spacecraft are bounded:

Assumption 7.3 *The disturbances acting on the follower spacecraft are bounded, i.e. there exist a positive constant β_{d_f} such that*

$$|d_f(t)| \leq \beta_{d_f}, \quad \forall t \geq t_0 \geq 0 \tag{7.10}$$

and that the difference between thrust and external disturbances acting on the leader spacecraft is bounded, that is:

$$|u_l(t) + d_l(t)| \leq \beta_{(u_l+d_l)}, \quad \forall t \geq t_0 \geq 0 \tag{7.11}$$

for a positive constant $\beta_{(u_l+d_l)}$.

We will also make use of the following assumption:

Assumption 7.4 *There exists some positive constant α_{r_l},α_{r_f}, such that $|r_l\,(t)| \geq \alpha_{r_l}$ and $|r_f\,(t)| \geq \alpha_{r_f}$ for all $t \geq t_0 \geq 0$.*

Since the leader spacecraft is assumed to follow an elliptic orbit, the assumption on the bound of r_l is plausible. The bound of r_f, may however seem inappropriate as the state vector (to be defined) implicitly depends on r_f. The choice of still making this assumption, is discussed in Remark 4.2.

Proposition 7.2 *Let $|\dot{\nu}_d\,(t_0)| \leq r_1$, $|\ddot{\nu}_d\,(t_0)| \leq r_2$, $\left|\dot{\tilde{\nu}}\,(t_0)\right| \leq r_3$ and $\left|\ddot{\tilde{\nu}}\,(t_0)\right| \leq r_4$ for some positive constants r_1, r_2, r_3 and r_4. Let Assumption 7.1-7.4 hold. Let the controller be given by*

$$u_f = m_f\ddot{\rho}_d + C_f\left(\dot{\nu}_d\right)\dot{\rho}_d + D_f\left(\dot{\nu}_d, \ddot{\nu}_d, r_l, \rho\right)\rho + n_f\left(r_l, \rho\right) - K_f\left(\dot{\rho}_0 - \dot{\rho}_r\right) \tag{7.12}$$

$$\dot{\rho}_r = \dot{\rho}_d - \Lambda_f e_f \tag{7.13}$$

$$\dot{\rho}_0 = \dot{\hat{\rho}} - \Lambda_f\tilde{\rho}, \tag{7.14}$$

where $\Lambda_f = \Lambda_f^\top \in \mathbb{R}^{3\times3}$ and $K_f := k_f I_{3\times3}$ are positive definite, $e_f := \rho - \rho_d \in \mathbb{R}^3$ is the position error and $\tilde{\rho} := \rho - \hat{\rho}$ is the observer estimation error. Let the observer be given by

$$\dot{\hat{\rho}} = a_f + (l_f I_{3\times3} + \Lambda_f)\,\tilde{\rho}_f \tag{7.15}$$

$$\dot{a}_f = \ddot{\rho}_d + l_f \Lambda_f \tilde{\rho}_f, \tag{7.16}$$

where $l_f > 2k_f/m_f$ scalar. With the input quantized, that is $\Psi(u_f)$, with $\Psi(\cdot)$ as in (7.4), the closed-loop system is UGPES with respect to Krasowskii solutions.

Proof. To ease the application of Theorem 7.1, we write the system on state space form as in (7.2), that is:

$$\dot{x}_1 = f_1(t, x_1) + g_1 u_1(t, x_1, \theta_1)$$

Here, $x_1 := (\dot{e}_f^\top, (\Lambda_f e_f)^\top, \dot{\tilde{\rho}}^\top, (\Lambda_f \tilde{\rho})^\top)^\top \in \mathbb{R}^{12}$ and $f_1(t, x_1) = (\sigma_1^\top, \sigma_2^\top, \sigma_3^\top, \sigma_4^\top)^\top$ with

$$\begin{aligned} \sigma_1 :=& -\ddot{\rho}_d - \frac{1}{m_f}\big\{C_f(\dot{\nu}_l)\dot{\rho} \\ & - D\left(\dot{\nu}_l, \ddot{\nu}_l, r_l, \rho\right)\rho - n_f(r_l, \rho) + d_f - \frac{m_f}{m_l}\left(u_l + d_l\right)\big\} && (7.17) \\ \sigma_2 :=& \Lambda_f \dot{e}_f && (7.18) \\ \sigma_3 :=& -\Lambda_f \dot{\tilde{\rho}} - l_f \dot{\tilde{\rho}} - l_f \Lambda_f \tilde{\rho} \\ & - \frac{1}{m_f}\big\{C_f(\dot{\nu}_l)\dot{\rho} - D_f\left(\dot{\nu}_l, \ddot{\nu}_l, r_l, \rho\right)\rho \\ & - n_f(r_l, \rho) + d_f - \frac{m_f}{m_l}\left(u_l + d_l\right)\big\} && (7.19) \\ \sigma_4 :=& \Lambda_f \dot{\tilde{\rho}} && (7.20) \end{aligned}$$

Furthermore,

$$g_1 := \left(1_{3\times3}, 0_{3\times3}, 1_{3\times3}, 0_{3\times3}\right)^\top, \tag{7.21}$$

and

$$u_1(t, x, \theta) = u_f \tag{7.22}$$

with u_f is as given in Proposition 7.2. Let us first define the Lyapunov function given by (cf. Berghuis and Nijmeijer (1993))

$$V_1(x_1) := \frac{1}{2} x_1^\top W^\top R_1 W x_1, \tag{7.23}$$

where $R_1 \in \mathbb{R}^{12\times 12}$ is defined as:

$$R_1 := \mathrm{diag}(m_f I_{3\times 3}, 2K_f\Lambda_f^{-1} - m_f I_{3\times 3}, m_f I_{3\times 3}, 2K_f\Lambda_f^{-1}) \tag{7.24}$$

and

$$W := \begin{bmatrix} I & I & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & I \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathbb{R}^{12\times 12}. \tag{7.25}$$

Note that for $K_f > m_f I_{3\times 3}\Lambda_f$, we have that

$$c_1 |x_1|^2 \leq V_1 \leq c_2 |x_1|^2$$

with $c_1 = \frac{1}{6}\lambda_{\min}(R_1)$ and $c_2 = \frac{3}{2}\lambda_{\max}(R_1)$, where $\lambda_{\min}(R_1) = m_f$ and $\lambda_{\max}(R_1) = 2k_f\lambda_{\max}(\Lambda_f)^{-1}$. This can be verified using the fact that $\frac{1}{3} \leq \lambda_{\min}(W^\top W)$ and that $\lambda_{\max}(W^\top W) \leq 3$. As shown in Proposition 4.2, (7.6) and (7.7) of Proposition 7.1 are satisfied with $x = x_1$, $p = 2$, $V_\delta = V_1$, $k_1 = \frac{1}{6}\lambda_{\min}(R_1) = \frac{1}{6}m_f$, $k_2 = \frac{3}{2}\lambda_{\max}(R_1(\delta)) = 3k_f^\star(\delta)/\lambda_{\max}(\Lambda_f)$ and $k_3(\delta) = \frac{1}{2}k_f^\star(\delta)$, where

$$\begin{aligned} k_f^\star :=& 2m_f\lambda_{\max}(\Lambda_f) + 4m_f\left(\beta_{\dot{\tilde{\nu}}} + \beta_{\dot{\nu}_d}\right) \\ &+ \frac{4}{\delta}\Big(\beta_{d_f} + \frac{m_f}{m_l}\beta_{(u_l+d_l)} \\ &+ 2m_f\beta_{\dot{\tilde{\nu}}}\beta_{\dot{\rho}_d} + m_f(\beta_{\dot{\tilde{\nu}}}^2 + 2\beta_{\dot{\tilde{\nu}}}\beta_{\dot{\nu}_d} + \beta_{\ddot{\tilde{\nu}}})\beta_\rho\Big) \end{aligned} \tag{7.26}$$

Furthermore,

$$\lim_{\delta\to 0}\frac{k_2(\delta)\delta^p}{k_1(\delta)} = \lim_{\delta\to 0}\frac{18k_f^\star(\delta)\delta^2}{\lambda_{\max}(\Lambda_f)m_f} = 0, \tag{7.27}$$

thus (7.9) is also satisfied. We will now prove (7.8) of Proposition 7.1. We have that

$$\frac{\partial V}{\partial x_1}g_1u_1 = x_1^\top W^\top R_1 W g_1 u_1. \tag{7.28}$$

By Assumption 7.1 and 7.4, we can therefore find positive constants β_{D_f}, β_{C_f} and β_{n_f}, defined as:

$$\beta_{D_f} = m_f\left(\frac{\mu}{\alpha_{r_f}} + \beta_{\dot{\nu}_d}^2\lambda_{\max}(\bar{D}) + \beta_{\ddot{\nu}_d}\lambda_{\max}(\bar{C})\right) \tag{7.29}$$

$$\beta_{C_f} = 2m_f\beta_{\dot{\nu}_d}\lambda_{\max}(\bar{C}) \tag{7.30}$$

$$\beta_{n_f} = m_f\mu\left(\frac{\beta_{r_l}}{\alpha_{r_f}^3} - \frac{1}{\alpha_{r_l}^2}\right) \tag{7.31}$$

such that $\beta_{D_f} \geq |D_f(\ddot{\nu}_d(t), \dot{\nu}_d(t), r_f(t))|$, $\beta_{C_f} \geq |C_f(\dot{\nu}_d(t))|$ and $\beta_{n_f} \geq |n_f(r_l(t), r_f(t))|$ for all $t \geq t_0 \geq 0$. This in turn, implies that the control law can be bounded as:

$$\begin{aligned} |u_f(t)| \leq & \beta_{n_f} + \beta_{D_f}\left(\frac{1}{\lambda_{\min}(\Lambda_f)}|x_1(t)| + \beta_{\rho_d}\right) \\ & + \beta_{C_f}\beta_{\dot{\rho}_d} + m_f\beta_{\ddot{\rho}_d} + k_f|x_1(t)|, \end{aligned} \tag{7.32}$$

for all $t \geq t_0 \geq 0$. We see that Assumption 7.4 were made to prevent u_f to grow unbounded. Now, let δ be a positive constant. Then, for any $|x_1| \geq \delta$ we have that

$$\frac{\partial V_1}{\partial x_1} g_1 u_1 \leq k_4(\delta)|x_1|^2 \tag{7.33}$$

where

$$\begin{aligned} k_4(\delta) = & 2m_f\Big\{\big(m_f\beta_{\ddot{\rho}_d} + \beta_{C_f}\beta_{\dot{\rho}_d} + \beta_{D_f}\beta_{\rho_d} \\ & + \beta_{n_f}\big)\frac{1}{\delta} + \beta_{D_f}\frac{1}{\lambda_{\min}(\Lambda_f)} + k_f(\delta)\Big\} \end{aligned} \tag{7.34}$$

where it has been used that $\lambda_{\max}(WRWg_1) = 2m_f$. Hence, the conclusion that the closed-loop system is UGPES on the chosen parameter set follows. ■

7.2.3 Simulations

In this section the performance of the proposed controller-observer scheme will be shown by simulations. The desired orbit of the leader spacecraft is of eccentricity $e_d = 0.5$, and has semimajor axis $a_d = 20000$ km. Both spacecraft are of mass $m_l = m_f = 100\,\text{kg}$. Furthermore, the thrust is assumed to be available in all directions of the leader spacecraft frame. The desired trajectory of the follower spacecraft is given by $\rho_d(t) = \text{col}(-10\cos\nu, 20\sin\nu, 0)$, which means that the follower spacecraft evolves around the leader spacecraft in an ellipse during their orbit around the Earth. The initial position and velocity of the follower spacecraft is chosen as $\rho(0) = \text{col}(-40, 20, 40)$ and $\dot{\rho}(0) = \text{col}(1, 0, -1)$, where as the initial states of the observer are $\hat{\rho}(0) = \text{col}(4, -4, 1)$ and $a_f(0) = \text{col}(-1, 4, 3)$. The controller and observer gains are as follows: $l_d = 0.5$, $K_f = 20I_{3\times3}$, $\Lambda_f = 0.06I_{3\times3}$.The logarithmic scheme of Proposition 7.2 is used with $u_0 = 2$ and $\rho = 0.2$. Figure 7.2 shows the position and velocity tracking error, and Figure 7.3 shows the actuation forces.

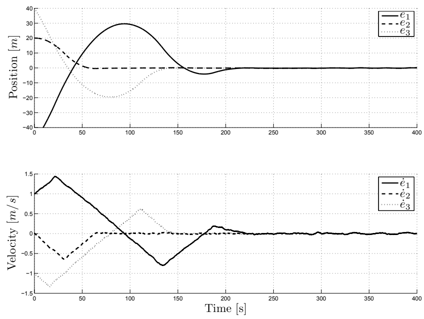

Figure 7.2: Position and velocity error with quantized actuation

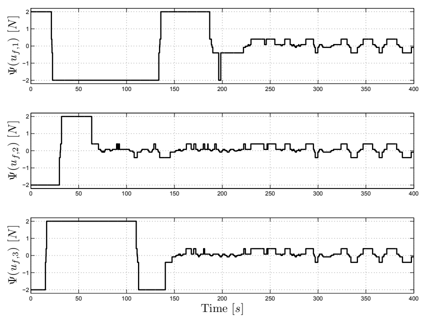

Figure 7.3: Force on follower spacecraft with quantized actuation

7.3 Pulse width modulated actuation

Commonly the actuation of spacecraft is pulse modulated. The fuel consumption is lowered; usually at the price of reduced accuracy. The stability analysis of such systems has mainly been focused on linear plants, see Gelig and Churilov (1998) and references therein, with a few exceptions including Hou and Michel (2001) and Teel et al. (2004). We will now show that although the control laws in the previous chapters were derived under the assumption of continuous actuation, similar stability properties are achievable even under pulse-width modulated systems. The analysis is based on Teel et al. (2004), where they consider systems of the following form:

$$\dot{x} = \varepsilon \left[f(x) + \sum_{i=1}^{m} g_i(x)\, u\left(h_i(x) - p_i(t)\right) \right], \tag{7.35}$$

where ε is a small positive parameter, $u : \mathbb{R} \to [0,1]$ is the unit step function with $u(0) = 1$, the functions $h_i : \mathbb{R}^n \to [0,1]$, f and g_i are continuous, and the functions $p_i : \mathbb{R} \to [0,1]$ are measurable, bounded and periodic with period one. The analysis of (7.35) can be reduced to the analysis of

$$\dot{x} \in f(x) + \sum_{i=1}^{m} g_i(x)\, S_i\left(h_i(x)\right), \tag{7.36}$$

since $\varepsilon > 0$ is small, and the state therefore changes slowly compared to p_i, so that the effect of p_i on (7.35) can be averaged. If, in addition $p_i(t)$ is chosen as $p_i(t) = t \bmod 1$, then (7.36) becomes simply,

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\, h_i(x), \tag{7.37}$$

Remark 7.2 *Even for systems of the form*

$$\dot{x} = f(x) + \sum_{i=1}^{k} g_{ci}(x)\, h_{ci}(x)$$

where the domain of h_{ci} is not necessarily $[0,1]$, but instead in $[-\underline{h}_{ci}, \overline{h}_{ci}]$, we can write the system in the form of (7.37). Just let $g_i = g_{cj}\overline{h}_j$ and $h_i = h_{cj}/\overline{h}_j$ for $i = 1, \cdots k$, $j = 1, \cdots, k$, and let $g_i = g_{cj}\underline{h}_j$ and $h_i = h_{cj}/\underline{h}_j$ for $i = k+1, \cdots, 2k = m$, $j = 1, \cdots, k$.

To keep this section self-contained, we will restate (Teel et al., 2004, Theorem1) here:

Theorem 7.2 (Teel et al., 2004, Theorem 1) *Suppose the functions f, g_i, h_i are continuous and that for (7.36) the compact set $\mathcal{A}$ is asymptotically stable with basin of attraction $\mathcal{H}$. Under these conditions, the set $\mathcal{H}$ is open and*

- *for each continuous function $w : \mathcal{H} \to \mathbb{R}_{\geq 0}$ that is positive definite with respect to $\mathcal{A}$ and proper with respect to $\mathcal{H}$, there exists $\beta \in \mathcal{KL}$,*
- *and, for each $\delta > 0$ and compact $\mathcal{K} \subset \mathcal{H}$, there exists $\varepsilon^* > 0$*

such that

$$\varepsilon \in (0, \varepsilon^*], \quad x(t_0) \in \mathcal{K} \implies$$

the (generalized Krasowskii/Filippov) solutions of (7.35) exist for all $t \geq t_0$ and satisfy

$$w(x(t)) \leq \beta\left(w(x(t_0)), \varepsilon(t - t_0)\right) + \delta, \quad \forall t \geq t_0.$$

In the proof of Theorem 7.2, asymptotic stability of the compact set $\mathcal{A}$ with basin of attraction $\mathcal{H}$ is used, by applying Theorem (Teel and Praly, 2000, Proposition 3), to ensure existence of a function $\beta_\circ \in \mathcal{KL}$, such that the solutions of system (7.36) that starts in $\mathcal{H}$ satisfy

$$w(x(t)) \leq \beta_\circ\left(w(x(0)), t\right), \quad \forall t \geq 0.$$

We will now give sufficient conditions for this hold:

Proposition 7.3 *Given any $\delta > 0$, suppose that there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, some class $\mathcal{K}$ functions $\underline{\alpha}$, $\overline{\alpha}$ and α such that for any $x \in \mathbb{R}^n \backslash \overline{\mathcal{B}}_\delta$, and any $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}\left(|x|\right) \leq V\left(x\right) \leq \overline{\alpha}\left(|x|\right) \tag{7.38}$$

$$\frac{\partial V}{\partial x}\left(x\right)\left(f\left(x\right) + \sum_{i=1}^{m} g_i\left(x\right) h_i\left(x\right)\right) \leq -\alpha\left(|x|\right) \tag{7.39}$$

Then, there exists a continuous function $w\left(\cdot\right) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, positive definite with respect to $\overline{\mathcal{B}}_{\tilde{\delta}}$ with $\tilde{\delta} := \overline{\alpha}\left(\delta\right)$ and proper with respect to $\mathbb{R}^n$, such that

$$w\left(|x\left(t\right)|\right) \leq \beta\left(w\left(|x_0|\right), t\right), \quad \forall t \geq 0,$$

for some function $\beta \in \mathcal{KL}$.

Proof. Assume that $V\left(x\right) \geq \overline{\alpha}\left(\delta\right)$ Then $|x| \geq \overline{\alpha}^{-1}\left(V\left(x\right)\right) \geq \delta$ such that

$$\dot{V} \leq -\tilde{\alpha}\left(V\right),$$

with $\tilde{\alpha}\left(\cdot\right) := \alpha \circ \overline{\alpha}^{-1}\left(\cdot\right)$. Now, define

$$|V|_{\tilde{\delta}} := V - \tilde{\delta}$$

Since $\tilde{\delta}$ is constant, for $V \geq \tilde{\delta}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{|V|_{\tilde{\delta}}\right\} = \dot{V} \leq -\tilde{\alpha}\left(V\right) \leq -\tilde{\alpha}\left(|V|_{\tilde{\delta}}\right),$$

(This holds trivially for $V < \tilde{\delta}$). By the comparison lemma,

$$|V\left(x\left(t\right)\right)|_{\tilde{\delta}} \leq \beta\left(|V\left(x\left(0\right)\right)|_{\tilde{\delta}}, t\right), \quad \forall t \geq 0,$$

with $\beta \in \mathcal{KL}$, and the conclusion follows by taking $w := |V|_{\tilde{\delta}}$. ■

As an illustration of the above result, we will again consider the control problem investigated in Chapter 6, but now with the actuation being pulse width modulated.

Proposition 7.4 *Consider the model (6.1-6.2), with control laws (6.3-6.6) and (6.9-6.12). Let Assumption 6.1 hold, and let $l_l \geq 2k_l/m_l$ and $k_l > 9/2k_l^\star$, where*

$$k_l^\star := m_l\ell_l + 4m_l\beta_{\dot{\nu}_o} + 2\frac{m_f}{\ell_l}\left(\beta_{\dot{\nu}_o}^2 + \beta_{\ddot{\nu}_o}\right) + 4\frac{\beta_d}{\delta}, \tag{7.40}$$

and let $l_f \geq 2k_f/m_f$ and $k_f > 3k_f^\star/2$, where

$$k_f^\star := m_f\ell_f + 4m_f\beta_{\dot{\nu}_o} + 2\frac{m_f}{\ell_l}\left(\beta_{\dot{\nu}_o}^2 + \beta_{\ddot{\nu}_o}\right) + 4\frac{\beta_d}{\delta} + \frac{3}{2}k_l. \qquad (7.41)$$

Assume that there exists a positive constant β_d, such that $|d(t)| \leq \beta_d$, for any $t \geq t_0 \geq 0$, where $d = (d_1^\top, d_2^\top)^\top$ with d_1 as defined in (6.15) and $d_2 = d_l$. Finally, assume that the control law is pulse width modulated with switching signal $p_i(t) = t \bmod 1$. Then, for each $\bar{\delta}$, and compact $\mathcal{K} \subset \mathbb{R}^n$, there exists an $\varepsilon^\star > 0$ and a $\tilde{\delta} > 0$, such that for any $\varepsilon \in (0, \varepsilon^\star]$, $x(t_0) \in \mathcal{K}$ the solutions exist for all $t \geq t_0$ and satisfy

$$|V(x(t))|_{\tilde{\delta}} \leq \beta\left(|V(x(t_0))|_{\tilde{\delta}}, \varepsilon(t - t_0)\right) + \bar{\delta}, \quad \forall t \geq 0, \qquad (7.42)$$

with V as in (6.20).

Proof. The model (6.1-6.2), with control laws (6.3-6.6) and (6.9-6.12) can be written on the form of (7.37), although with f and h also depending on t. It is the choice $p_i(t) = t \bmod 1$ which simplifies the analysis to the problem of finding an asymptotically stable compact set $\mathcal{A}$ with basin of attraction $\mathcal{H}$ for (7.37). We showed in the proof of Proposition 6.1, that (7.38) holds with $V(x)$ as in (6.20), with $\underline{\alpha}(s) = \frac{1}{6}\min\{m_l, m_f\} s^2$, $\overline{\alpha}(s) = 3\max\{k_l/\ell_l, k_f/\ell_f\} s^2$. Given any $\delta > 0$, the time derivative of $V(x)$ can, by using (6.21), $|x_1||x_2| \leq |x_1|^2 + |x_2|^2$, $k_l|x_1||x_2| \leq k_l/3|x_2|^2 + 3k_l/4|x_1|^2$ and that $|d(t)| \leq \beta_d$, for all $t \geq t_0$, be bounded as,

$$\begin{aligned}\dot{V} \;\leq\; & -\left(k_f - m_f\ell_f - 4m_f\beta_{\dot{\nu}_o} - 2\frac{m_f}{\ell_l}\left(\beta_{\dot{\nu}_o}^2 + \beta_{\ddot{\nu}_o}\right) - \frac{3}{2}k_l - 4\frac{\beta_d}{\delta}\right)|x_1|^2 \\ & -\left(k_l\left(1 - \frac{2}{3}\right) - m_l\ell_l - 4m_l\beta_{\dot{\nu}_o} - 2\frac{m_f}{\ell_l}\left(\beta_{\dot{\nu}_o}^2 + \beta_{\ddot{\nu}_o}\right) - 4\frac{\beta_d}{\delta}\right)|x_2|^2,\end{aligned}$$

for any $x \in \mathbb{R}\backslash\overline{\mathcal{B}}_\delta$. Using $k_l > 9/2k_l^\star$ and $k_f > 3k_f^\star/2$, we get that (7.39) holds with $\alpha(s) = -1/2k^\star s^2$. By Proposition 7.3, we find that

$$|V(x(t))|_{\tilde{\delta}} \leq \beta\left(|V(x(0))|_{\tilde{\delta}}, t\right), \quad \forall t \geq 0.$$

The conclusion follows by applying Theorem 7.2; see the discussion below Theorem 7.2. ■

Remark 7.3 *Using the bounds for $V(x)$, namely $\underline{\alpha}(|x|) = \frac{1}{6}\min\{m_l, m_f\}|x|^2$, $\overline{\alpha}(|x|) = 3\max\{k_l/\ell_l, k_f/\ell_f\}|x|^2$, the inequality 7.42 can be rewritten as:*

$$|x(t)| \leq \beta^\star(|x_0|, \varepsilon(t - t_0)) + \delta^\star,$$

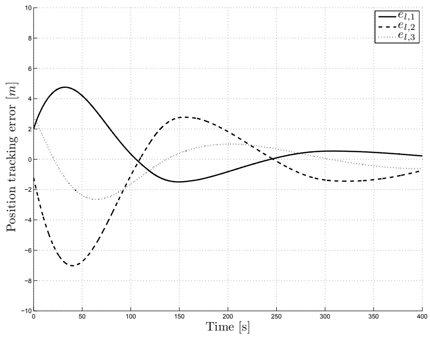

Figure 7.4: Position tracking error of leader spacecraft

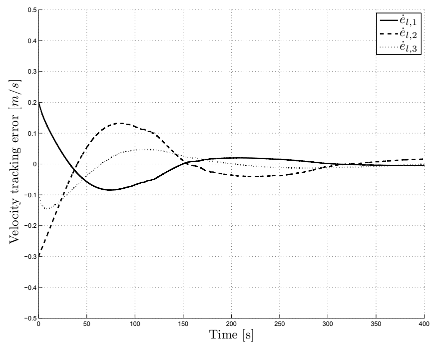

Figure 7.5: Velocity tracking error of leader spacecraft

where

$$\beta^{\star}(s,t) := \underline{\alpha}^{-1}\left(\beta\left(\overline{\alpha}(s), t\right)\right) \in \mathcal{KL}$$

and

$$\delta^{\star} := \left(\frac{6\left(\tilde{\delta}+\bar{\delta}\right)}{\min\left\{m_l, m_f\right\}}\right)^{\frac{1}{2}}.$$

7.3.1 Simulations

As an illustration, we study the same problem as in Section 6.6, except that our control law is now pulse-width-modulated, and the disturbance is the J_2 disturbance described in Section 3.3.1, instead of the impact disturbances. Figure 7.4 to 7.7 show the position tracking error, velocity tracking error, position estimation error and velocity estimation error, respectively, of the leader spacecraft. Figure 7.8 shows the control forces acting on the leader spacecraft, and Figure 7.9 shows the same over a shorter time period. The corresponding figures for the follower spacecraft are given in 7.10 to 7.15.

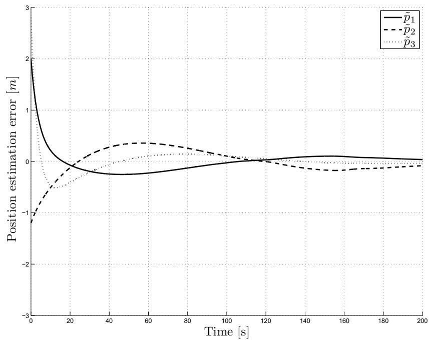

Figure 7.6: Position estimation error of leader spacecraft

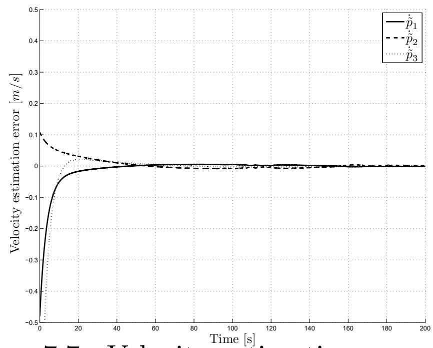

Figure 7.7: Velocity estimation error of leader spacecraft

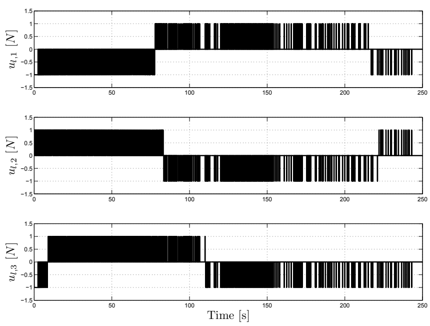

Figure 7.8: Control forces acting on leader spacecraft

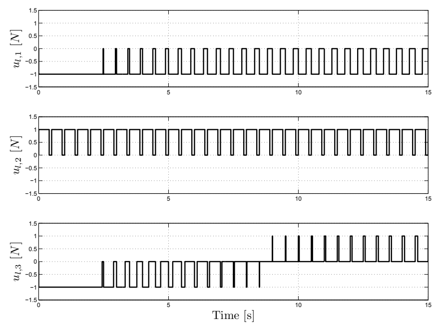

Figure 7.9: Control forces acting on leader spacecraft

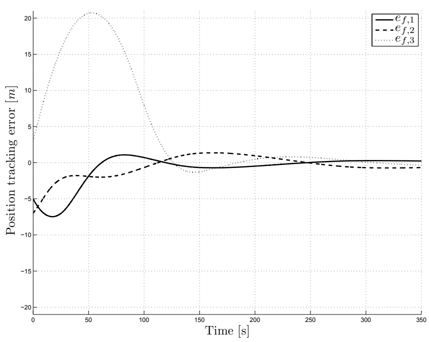

Figure 7.10: Position tracking error of follower spacecraft

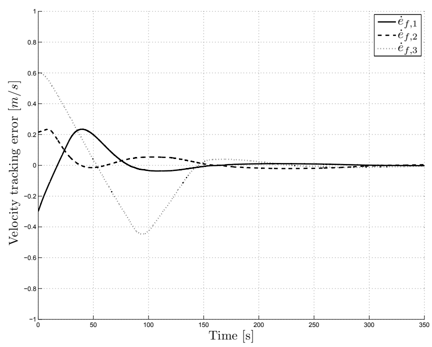

Figure 7.11: Velocity tracking error of follower spacecraft

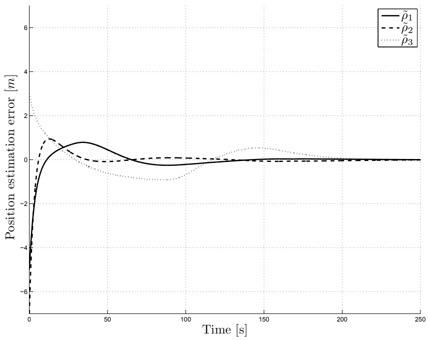

Figure 7.12: Position estimation error of follower spacecraft

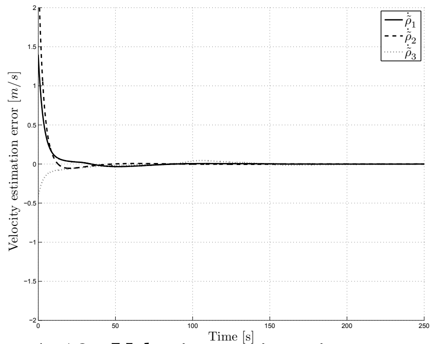

Figure 7.13: Velocity estimation error of follower spacecraft

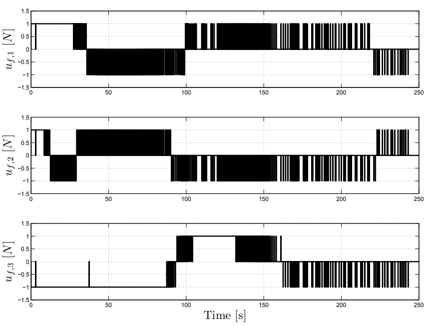

Figure 7.14: Control forces acting on follower spacecraft

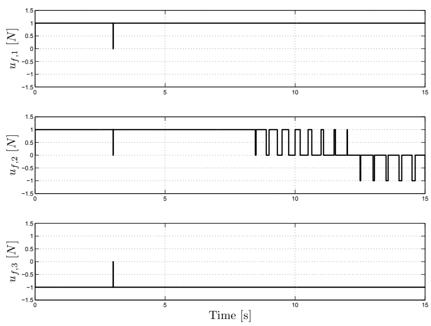

Figure 7.15: Control forces acting on follower spacecraft

Chapter 8

Conclusion and future work

8.1 Conclusion

This thesis has been concerned with the problem of robust stability of nonlinear control algorithms applied to spacecraft in formation. By robustness we mean for instance the ability of a system to decrease the steady state offset under perturbations, by appropriately choosing the tuning parameters. This criteria has formally been stated in the new definition of uniform global practical exponential stability (UGPES), and similar semiglobal definitions. For ease of application, we have provided Lyapunov sufficient conditions for a system to satisfy the definitions, both for ordinary systems and interconnected system on a cascaded structure. The mathematical framework were further extended by considering a class of disturbances, to which solutions of input-to-state stable systems would show uniform asymptotic stability properties (outside a ball of the origin of the nominal system, which corresponds to a predefined level of accuracy). These mathematical tools, were applied in the reminder of the thesis to analyse control algorithms for spacecraft in leader-follower formation.

To serve the need of being cost- and space efficient, we have proposed algorithms, both for rotational and translational tracking, which have low demands to measurements and intervehicle communication. In fact, we have proven that in the translational case, UGPES can be achieved when only absolute position is available to both spacecraft, and relative position is available to the follower spacecraft. In the rotational case UPES (local result) is achieved when the relative attitude to the inertial frame is available to the leader spacecraft, and the relative attitude between the spacecraft are available to the follower. In both cases closely connected controller-observer algorithms were used, which are usually more compact than when the controller design is separated from the observer design. This concludes the research done on attitude tracking of leader-follower spacecraft formation in this thesis, and the focus was again directed toward relative translation.

We then proceeded with removing the assumption that the disturbance is bounded in its norm. Instead we assume that the disturbance belong to a certain class with limited excitation in average. We show that the proposed controller-observer for the system is input-to-state stable, and that we are able to identify certain measures, such as precision and convergence rate, depending on the disturbance excitation over a certain time window.

The above described work, has assumed that continuous actuation could be provided by the spacecraft. To extend the area of application, we have considered two approaches in this thesis: quantized and pulse width modulated actuation.

8.2 Recommendations for future work

8.2.1 Qualitative behavior of ISS systems

Based on the results in Section 2.5, one should exploit the possibility to tune some control gains in order to enlarge the class of signals to which the systems is robust or, equivalently, to increase the precision for signals belonging to a given class, and make the connection to practical stability. Another possibility would be to achieve similar properties for integral ISS systems.

8.2.2 Formations about other planets and in deep space

The leader follower formations described in this thesis were intended for Earth orbiting applications. Most of the results, could possibly be applied to formations around other planets or even in deep space, with only minor modifications. For example will the models for a quasi-Halo orbit, as described in e.g. Di Giamberardino and Monaco (1997), be of a similar structure as the models presented in this thesis.

8.2.3 Internal model approach to robust control

Based on the recent developments in Zhang and Serrani (2006) and Zhang and Serrani (2009), the results of Serrani (2003) for circular orbits, could possibly be extended to elliptical orbits. The internal model approach is somewhat unrelated to the control design approaches taken in this thesis, but non the less interesting as robustness properties desirable for spacecraft formation are typically achieved. In addition the formation type considered in Serrani (2003), is practically the same as in this thesis.

8.2.4 Quantized and pulse modulated control

In Chapter 7 of this thesis, we consider control with quantized inputs. The results are based on the use of a logarithmic quantizer, which is not very useful for practical purposes as it requires infinite valued feedback laws. The results however, may be improved by considering the truncated version of the logarithmic quantizer, as described Ceragioli and De Persis (2007). This finite valued feedback law could be used to achieve semiglobal practical stability of the overall formation. For the analysis in Chapter 7 to be more realistic, it would be appropriate to include the actuator configurations for the spacecraft. In the author's opinion it would also be interesting to investigate stability of a pulse modulated system using the recent developments in hybrid systems analysis, see e.g. Goebel et al. (2009) and references therein.

8.2.5 Attitude tracking

As discussed in Section 1.3.2, perfect tracking in term of the quaternion error q_{dl}, is achieved when $q_{dl} = (\pm 1, 0, 0, 0)$. In Chapter 5, the proposed control schemes only achieve stability with respect to one of the equilibrium points. Therefore, two local controller could be used, as in Kristiansen et al. (2009), one for each equilibrium point. To account for measurement noise while avoiding infinite switching between the controllers, the work of Prieur et al. (2007) should be considered. Also, future work should aim at increasing the domain of attraction to allow for greater angular estimation- and tracking velocity errors.

8.2.6 Multispacecraft formations

As shown in Ploen et al. (2004b) (see also Ploen et al. (2004a)), the models of Chapter 3 can be conveniently stacked into a model with the same basic properties. This means that most of the control algorithms of this thesis also can be extended to multispacecraft formations. The stacked model can also be used for other types of formations, such as virtual structure formations, by not describing the relative motion of each spacecraft to a leader, but rather the relative motion of each spacecraft to a convenient point of reference. Considering multispacecraft formation will however introduce complexity in several aspects; guidance algorithms, communication topology and collision avoidance are a few examples.

8.2.7 Communication delay

Although there has been focused on keeping the need for intervehicle communication at a low level, due to limited bandwidth, all signals considered in this thesis are deterministic and all communication is without delay. These are topics that should be addressed.

8.2.8 External disturbances and numerical issues in simulation

The control algorithms should be tested with realistic disturbances, by using one of the commercially available packages for MATLAB, e.g. the Spacecraft Control Toolbox from Princeton Satellite Systems[1], which has its own formation flying module.

The rigid body dynamics should also be considered simulated using the method outlined in e.g. Celledoni and Säfström (2006), as it yields a faster and more important, more accurate numerical solution compared to standard numerical solvers.

8.2.9 Other topics

There are several topics which were not handled at all in this thesis, but are still closely related, see Section 1.2.3. For example has this thesis not dealt with the problem of finding appropriate trajectories for the formation. Such trajectories should be fuel efficient, but also be ideal for the formation task in question. The Clohessy-Wiltshire equation, Clohessy and Wiltshire (1960), originally stated for orbital rendezvous and considered in several papers on formation flying, are not particularly suited from a fuel efficiency point of view as they do not take into account the oblateness of Earth. A realistic setup for the control algorithms proposed in this thesis, would rather build on Schaub and Alfriend (2001), were an analytic method for establishing J_2 invariant relative orbits is presented.

[1] For more information see http://www.psatellite.com. Accessed 29th of October, 2009

Bibliography

Angeli, D., Nešić, D., 2001. Power characterizations of input-to-state stability and integral input-to-state stability. IEEE Transactions on Automatic Control 48, 1298–1303.

Angeli, D., Sontag, E. D., Wang, Y., 2000. A characterization of integral input-to-state stability. IEEE Transactions on Automatic Control 45, 1082–1097.

Benabdallah, A., Ellouze, I., Hammami, M. A., 2009. Practical stability of nonlinear time-varying cascade systems. Journal of Dynamical and Control Systems 15, 45–62.

Berghuis, H., Nijmeijer, H., 1993. A passivity approach to controller-observer design for robots. IEEE Transactions on Robotics and Automation 9 (6), 740–754.

Berghuis, H., Nijmeijer, H., October 1994. Robust control of robots via linear estimated state feedback. IEEE Transactions on Automatic Control 39 (10), 2159–2162.

Bondhus, A. K., Pettersen, K. Y., Gravdahl, J. T., 2005. Leader/follower synchronization of satellite attitude without angular velocity measurements. In: Proc. of 44th IEEE Conference on Decision & Control, and the European Control Conference. pp. 7270–7277.

Børhaug, E., 2008. Nonlinear control and synchronization of mechanical systems. Ph.D. thesis, Norwegian University of Science and Technology, ISBN:978-82-471-1298-4.

Caccavale, F., Villani, L., 1999. Output feedback control for attitude tracking. System and Control Letters 38, 91–98.

Celledoni, E., Säfström, N., 2006. A symmetric splitting method for rigid body dynamics. Modeling, Identification and Control 27, 95–108.

Ceragioli, F., 1999. Discontinuous ordinary differential equations and stabilization. Ph.D. thesis, Universitá degli Studi di Firenze.

Ceragioli, F., De Persis, C., 2007. Discontinuous stabilization of nonlinear systems: Quantized and switching controls. Systems and Control Letters 56, 461–473.

Chaillet, A., 2006. On stability and robustness of nonlinear systems to cascaded systems. Ph.D. thesis, Université Paris Sud - LSS - Supélec.

Chaillet, A., Loría, A., 2006a. Necessary and sufficient conditions for uniform semiglobal practical asymptotic stability: Application to cascaded systems. Automatica 42, 1899–1906.

Chaillet, A., Loría, A., 2006b. Uniform global practical asymptotic stability for non-autonomous cascaded systems. European Journal of Control 12 (6), 595–605.

Chaillet, A., Loría, A., 2006c. Uniform global practical stability for non-autonomous cascaded systems. In: Proc. of Int. Symp. on Mathematical Theory of Networks and Systems.

Chaillet, A., Loría, A., 2008. Uniform semiglobal practical asymptotic stability for non-autonomous cascaded systems and applications. Automatica 44, 337–347.

Chellaboina, V., Haddad, M., 2002. A unification between partial stability and stability theory for time-varying systems. IEEE Control Systems Magazine, 66–75.

Clohessy, W. H., Wiltshire, R. S., 1960. Terminal guidance system for satellite rendezvous. Journal of Aerospace Sciences 27, 9.

Corless, M., Glielmo, L., 1998. New converse lyapunov theorems and related results on exponential stability. Mathematics of Control Signals, and Systems 11, 79–100.

Cortés, J., 2008. Discontinuous dynamical systems: A tutorial on solutions, nonsmooth analysis, and stability. IEEE Control Systems Magazine 28, 36–73.

D'Amico, S., Montenbruck, O., Arbinger, C., Fiedler, H., 2005. Formation flying concept for close remote sensing satellites.

Di Giamberardino, P., Monaco, S., 1997. On halo orbits spacecraft stabilization. Acta Astronautica 38, 903–925.

Dlala, M., Hammami, M. A., 2007. Uniform exponential practical stability of impulsive perturbed systems. Journal of Dynamical and Control Systems 13, 373–386.

Egeland, O., Godhavn, J., 1994. Passivity-based adaptive attitude control of rigid spacecraft. IEEE Transactions on Automatic Control 39, 842–846.

Egeland, O., Gravdahl, J. T., 2002. Modeling and Simulation for Automatic Control. Marine Cybernetics, ISBN: 82–92356-00-2.

Egorov, V. A., 1958. Definition of true anomaly in perturbed motion. Soviet Astronomy 2, 147–149.

Filippov, A. F., 1988. Differential Equations with Discontiuous Righthand Sides. Kluwer Academic Publishers, ISBN:90-277-2699-X.

Fossen, T. I., 2002. Marine Control Systems: Guidance, Navigation, and Control of Ships, Rigs and Underwater Vehicles. Marine Cybernetics.

Fragopoulos, D., Innocenti, M., 2004. Stability considerations in quaternion attitude control using discontinuous Lyapunov functions. IEE Proc. Control Theory Applications 151.

Gelig, A. K., Churilov, A. N., 1998. Stability and Osciallations of Nonlinear Pulse-Modulated System. Birkhäuser.

Gill, E., Steckling, M., Butz, P., 2001. Gemini: A milestone towards autonomous formation flying. In: Proc. of the ESA Workshop on On-Board Autonomy.

Goebel, R., Sanfelice, R. G., Teel, A. R., 2009. Hybrid dynamical systems: Robust stability and control for systems that combine continuous-time and discrete-time dynamics. IEEE Control Systems Magazine, 28–93.

Gonzalez, H., Grøtli, E. I., Templeton, T. R., Biermeyer, J. O., Sprinkle, J., Sastry, S. S., 2008. Transitioning control and sensing technologies from fully-autonomous driving to driver assistance systems.

In: Automatisierungs-, Assistenzsysteme und eingebettete Systeme für Transportmittel.

Grøtli, E. I., 2007. Analysis of a nonlinear continuous control algorithm, in the case of discontinuous actuation. In: Proc. of the 58th International Astronautical Congress.

Grøtli, E. I., Chaillet, A., Gravdahl, J. T., 2008. Output control of spacecraft in leader follower formation. In: Proc. of the 47th IEEE Conference on Decision and Control. pp. 1030–1035.

Grøtli, E. I., Chaillet, A., Panteley, E., Gravdahl, J. T., 2010a. Robustness of ISS systems to inputs with limited moving average, with application to spacecraft formations. International Journal of Robust and Nonlinear Control(Submitted).

Grøtli, E. I., Chaillet, A., Panteley, E., Gravdahl, J. T., 2010b. Robustness of ISS systems to inputs with limited moving average, with application to spacecraft formations. In: Proc. of the 7th International Conference on Informatics in Control, Automation and Robotics. (Submitted).

Grøtli, E. I., Gravdahl, J. T., 2007. Passivity based controller-observer schemes for relative translation of a formation of spacecraft. In: Proc. of the 26th American Control Conference. pp. 4684–4689.

Grøtli, E. I., Gravdahl, J. T., 2008a. Formation control by quantized output feedback. In: Proc. of the 3rd International Symposium on Formation Flying, Missions and Technologies.

Grøtli, E. I., Gravdahl, J. T., 2008b. Output attitude tracking of formation of spacecraft. In: Proc. of the 17th IFAC World Congress. pp. 2137–2142.

Grüne, L., 2002. Input-to-state dynamical stability and its Lyapunov function characterization. IEEE Transactions on Automatic Control 47, 1499–1504.

Grüne, L., 2004. Quantitative aspects of the input-to-state-stability property. In: de Queiroz, M., Malisoff, M., Wolenski, P. (Eds.), Optimal Control, Stabilization and Nonsmooth Analysis. Springer-Verlag, pp. 215–230.

Hahn, W., 1967. Stability of motion. Springer, ISBN: 978-0387038292.

Hanslmeier, A., Denkmayr, K., Weiss, P., 1999. Longterm prediction of solar activity using the combined method. Solar Physics 184, 213–218.

Hou, L., Michel, A. N., 2001. Stability analysis of pulse-width-modulated feedback systems with nonlinear plants. In: Proc. of the American Control Conference.

Hughes, P. C., 1986. Spacecraft Attitude Dynamics. John Wiley and Sons, Inc., ISBN: 978-0486439259.

Joshi, S. M., Kelkar, A. G., Wen, J. T.-Y., 1995. Robust attitude stabilization of spacecraft using nonlinear quaternion feedback. IEEE Transactions on Automatic Control 40, 1800–1803.

Khalil, H. K., 2002. Nonlinear Systems, 3rd Edition. Prentice Hall, ISBN: 0-13-067389-7.

Kokotoviѐ, P. V., Marino, R., 1986. On vanishing stability regions in nonlinear systems with high-gain feedback. IEEE Transactions on Automatic Control 31 (10), 967–970.

Kristiansen, R., Grøtli, E. I., Nicklasson, P. J., Gravdahl, J. T., 2007. A model of relative translation and rotation in a leader-follower spacecraft formations. Modeling, Identification and Control 28 (1), 3–13.

Kristiansen, R., Loría, A., Chaillet, A., Nicklasson, P. J., 2006a. Adaptive output feedback control of spacecraft relative translation. In: Proc. of the 45th Conference on Decision & Control. pp. 6010–6015.

Kristiansen, R., Loría, A., Chaillet, A., Nicklasson, P. J., 2006b. Output feedback control of relative translation in a leader-follower spacecraft formation. In: Pettersen, K. Y., Gravdahl, J. T., Nijmeijer, H. (Eds.), Group coordination and cooperative control. Springer-Verlag, pp. 131–151.

Kristiansen, R., Loría, A., Chaillet, A., Nicklasson, P. J., 2009. Spacecraft relative rotation tracking without angular velocity measurements. Automatica 45, 750–756.

Kristiansen, R., Nicklasson, P. J., Gravdahl, J. T., 2006c. Quaternion-based backstepping control of relative attitude in a spacecraft formation. In: Proc. of 45th IEEE Conference on Decision & Control. pp. 5724–5729.

Krogstad, T. R., Kristiansen, R., Gravdahl, J. T., Nicklasson, P. J., 2007. PID+ backstepping control of relative spacecraft attitude. In: Proc. of 7th IFAC Symposium on Nonlinear Control Systems.

Lin, Y., 1992. Lyapunov function techniques for stabilization. Ph.D. thesis, Rutgers, The State University of New Jersey.

Lin, Y., Sontag, E. D., Wang, Y., 1996. A smooth converse lyapunov theorem for robust stability. SIAM Journal on Control and Optimization 34, 124–160.

Lizarralde, F., Wen, J. T., 1996. Attitude control without angular velocity measurement: A passivity approach. IEEE Transactions on Automatic Control 41, 468–472.

Loría, A., Panteley, E., 2002. Uniform exponential stability of linear time-varying systems: revisited. Systems & Control Letters 47, 13–24.

Loría, A., Panteley, E., 2005. Cascaded nonlinear time-varying systems: Analysis and design. In: Lamnabhi-Lagarrigue, F., Loría, A., Panteley, E. (Eds.), Advanced Topics in Control Systems Theory. Springer-Verlag, pp. 23–64, ISBN: 9780387008936.

Loría, A., Panteley, E., 2006. Stability, told by its developers. In: Loría, A., Lamnabhi-Lagarrigue, F., Panteley, E. (Eds.), Advanced Topics in Control Systems Theory. Springer-Verlag, pp. 199–258.

Ma, Y., Soatto, S., Košecká, J., Sastry, S. S., 2004. An Invitation to 3-D Vision. From Images to Geometric Models. Vol. 26 of Interdisciplinary Applied Mathematics. Springer.

Montenbruck, O., Gill, E., 2000. Satellite Orbits: Models, Methods and Applications. Springer, ISBN: 3-540-67280.

NASA, 1999. On space debris. Tech. rep., NASA, ISBN: 92-1-100813-1.

Overhage, C. F. J., Radford, W. H., 1964. The Lincoln Laboratory West Ford Program - An historical perspective. Proceeding of the IEEE 52, 452–454.

Paden, B., Panja, R., 1988. Globally asymptotically stable 'PD+' controller for robot manipulators. International Journal of Control 47 (6), 1697–1712.

Pan, H., Kapila, V., 2001. Adaptive nonlinear control for spacecraft formation flying with coupled translational and attitude dynamics. In: Proc. of the 40th Conference on Decision & Control. pp. 2057–2062.

Panteley, E., Lefeber, E., Loria, A., Nijmeijer, H., 1998. Exponential tracking control of a mobile car using a cascaded approach. In: IFAC Workshop on Motion Control. pp. 221–226.

Panteley, E., Loría, A., 2001. Growth rate conditions for uniform asymptotic stability of cascaded time-varying systems. Automatica 37, 453–460.

Panteley, E., Loría, L., 1998. On global uniform asymptotic stability of nonlinear time-varying systems in cascade. Systems Control Letters 33, 131–138.

Pepe, P., 2007. On Liapunov-Krasovskii functionals under Carathéodory conditions. Automatica 43, 701–706.

Persson, S., Bodin, P., Gill, E., Harr, J., Jörgensen, J., 2006. Prisma - an autonomous formation flying mission. In: Proc. of the ESA Small Satellite Systems and Services Symposium.

Ploen, S. R., Hadaegh, F. Y., Scharf, D. P., 2004a. Rigid body equations of motion for modeling and control of spacecraft formations - part 1: Absolute equations of motion. In: Proc. of the American Control Conference. pp. 3646–3653.

Ploen, S. R., Scharf, D. P., Hadaegh, F. Y., Acikmese, A. B., 2004b. Dynamics of earth orbiting formations. In: Proc. of AIAA Guidance, Navigation and Control Conference.

Praly, L., Wang, Y., 1996. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability. Mathematics of Control, Signals, and Systems 9, 1–33.

Prieur, C., Goebel, R., Teel, A. R., 2007. Hybrid feedback control and robust stabilization of nonlinear systems. IEEE Transactions on Automatic Control 52, 2103–2117.

Pugh, C. C., 2002. Real Mathematical Analysis. Springer, ISBN:0-387-95297-7.

Queiroz, M. d., Yan, Q., Yang, G., Kapila, V., December 1999. Global output feedback tracking control of spacecraft formation flying with parametric uncertainty. In: Proc. of the 38th Conference on Decision & Control. Phoenix, Arizona, pp. 584–589.

Salcudean, S., 1991. A globally convergent angular velocity observer for rigid body motion. IEEE Transactions on Automatic Control 36, 1493–1497.

Sastry, S., 1999. Nonlinear Systems: Analysis, Stability and Control. Springer, ISBN: 0-387-98513-1.

Sastry, S., Bodson, M., 1994. Adaptive Control: Stability, Convergence, and Robustness. Prentice-Hall, ISBN: 0-13-004326-5.

Scharf, D. P., Hadaegh, F. Y., Ploen, S. R., 2003. A survey of spacecraft formation flying guidance and control (Part I): Guidance. Proceeding of the American Control Conference, 1733–1739.

Scharf, D. P., Hadaegh, F. Y., Ploen, S. R., 2004. A survey of spacecraft formation flying guidance and control (Part II): Control. In: Proc. of the 2004 American Control Conference. pp. 2976– 2985.

Schaub, K., Alfriend, K. T., 2001. j_2 invariant relative orbits for spacecraft formations. Celestial Mechanics and Dynamical Astronomy 79, 77–95.

Sellers, J. J., 2000. Understanding Space: An Introduction to Astronautics. The McGraw-Hill Companies, Inc., ISBN: 0-07-242468-0.

Sepulchre, R., 2000. Are basins of attraction easy to enlarge by feedback? Plenary lecture at Congress on theoretical and applied mechanics.

Serrani, A., 2003. Robust coordinated control of satellite formations subject to gravity perturbations. In: Proc. of the 2003 American Control Conference. pp. 302–307.

Skjetne, R., 2005. The maneuvering problem. Ph.D. thesis, Norwegian University of Science and Technology, ISBN:82-471-6861-8.

Sontag, E. D., 1989. Smooth stabilization implies coprime factorization. IEEE Transactions on Automatic Control 34, 435–443.

Sontag, E. D., 1998. Comments on integral variants of ISS. Systems & Control Letters 34, 93–100.

Sontag, E. D., 2007. Input to state stability: Basic concepts and results. In: Nistri, P., Stefani, G. (Eds.), Nonlinear and Optimal Control Theory. Springer-Verlag, Berlin, pp. 163–220.

Sontag, E. D., Wang, Y., 1995. On characterizations of the input-to-state stability property. Systems & Control Letters 24, 351–359.

Sprinkle, J., Eklund, J. M., Gonzalez, H., Grøtli, E. I., Sanketi, P., Moser, M., Sastry, S. S., 2010. Recovering models of a four-wheel vehicle using vehicular system data. In: -. (In preparation).

Sprinkle, J., Eklund, J. M., Gonzalez, H., Grøtli, E. I., Upcroft, B., Makarenko, A., Uther, W., Moser, M., Fitch, R., Durrant-Whyte, H., Sastry, S. S., 2009. Model-based design: a report from the trenches of the DARPA Urban Challenge. Software and Systems Modeling 8, 551–566.

Sørdalen, O. J., Egeland, O., 1995. Exponential stabilization of nonholonomic chained systems. IEEE Transactions on Automatic Control 40, 35–49.

Tayebi, A., 2006. Unit quaternion observer based attitude stabilization of a ridgid spacecraft without velocity measurements. In: Proc. of 45th IEEE Conference on Decision & Control. pp. 1557–1561.

Teel, A., Panteley, E., Loría, A., 2002. Integral characterizations of uniform asymptotic and exponential stability with applications. Mathematics of Control, Signals, and Systems 15, 177–201.

Teel, A. R., Moreau, L., Nešić, D., 2004. Input to state stability of pulse width modulated control systems. Systems Control Letters 51, 23–32.

Teel, A. R., Praly, L., 2000. A smooth lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite function. ESAIM 5, 313–367.

Tjønnås, J., 2008. Nonlinear and adaptive dynamic control allocation. Ph.D. thesis, Norwegian University of Science and Technology, ISBN: 978-82-471-7250-6.

Tjønnås, J., Chaillet, A., Panteley, E., Johansen, T. A., 2006. Cascade lemma for set-stable systems. In: Proc. of the 45th IEEE Conference on Decision & Control. pp. 338–342.

Vallado, D. A., 2001. Fundamentals of Astrodynamics and Applications, 2nd Edition. Kluwer Academic Publishers, ISBN: 978-1881883142.

Vorotnikov, V. I., 1998. Partial Stability and Control. Birkhäuser, ISBN: 0-8176-3917-9.

Wertz, J. R. (Ed.), 1978. Spacecraft attitude determination and control. D. Reidel Publishing company, ISBN: 9027709599.

Wong, H., Pan, H., Kapila, V., 2005. Output feedback control for spacecraft formation flying with coupled translation and attitude dynamics. In: Proc. of 26th American Control Conference. pp. 2419–2426.

Wong, H., Pan, H., Queiroz, M. S. d., Kapila, V., 2001. Adaptive learning control for spacecraft formation flying. In: Proc. of the 40th Conference of Decision & Control. pp. 1089–1094.

Xu, Y., N., F.-C., Lind, R., Tatsch, A., 2007. μ control for satellites formation flying. Journal of Aerospace Engineering 20, 10–21.

Yan, Q., Yang, G., Kapila, V., Queiroz, M. S. d., 2000. Nonlinear dynamics and output feedback control of multiple spacecraft in elliptical orbits. In: Proc. of the 2000 American Control Conference. pp. 839–843.

Zhang, Z., Serrani, A., 2006. The linear periodic output regulation problem. Systems Control Letters 55, 518–529.

Zhang, Z., Serrani, A., 2009. Adaptive robust output regulation of uncertain linear periodic systems. IEEE Transactions on Automatic Control 54, 266–278.

Appendix A

Proofs of UGPES and UGPAS for systems on cascade

A.1 UGPES

Proof of Theorem 2.5. Due to Young's inequality and Assumption 2.3 we can find the following bound:

$$\begin{aligned}|x_1|\,\sigma\left(x_1,x_2,\theta_2^\star\right) \le{}& |x_1| + |\theta_2^\star|\,|x_1|^2 + \frac{1}{2}\left(|x_1|^2+|x_2|^2\right)\\&+|\theta_2^\star|\,\frac{1}{2}\left(\lambda\,|x_1|^2+\frac{1}{\lambda}\,|x_2|^2\right)\\={}&|x_1|+\frac{1}{2}\left(1+(2+\lambda)\,|\theta_2^\star|\right)|x_1|^2\\&+\frac{1}{2}\left(1+\frac{1}{\lambda}\,|\theta_2^\star|\right)|x_2|^2,\end{aligned}$$

where λ is a constant. Now, define

$$\mathcal{V} := V_{\delta_1}+V_{\delta_2}.$$

The time derivative of $\mathcal{V}$ along (2.19) and (2.20) is:

$$\begin{aligned}\dot{\mathcal{V}}\le{}&-\left(\kappa_1-\frac{\tilde c}{|x_1|}-\tilde c\frac{1}{2}\left(1+(2+\lambda)\,|\theta_2^\star|\right)\right)|x_1|^2\\&-\left(\kappa_2-\frac{c_2}{|x_2|}-\tilde c\frac{1}{2}\left(1+\frac{1}{\lambda}\,|\theta_2^\star|\right)\right)|x_2|^2\end{aligned}$$

Now, define

$$\delta := \max\{\delta_1, \delta_2\},$$

and notice that

$$|x| \geq \delta \implies \max\{|x_1|, |x_2|\} \geq \frac{\delta}{\sqrt{2}}.$$

We will now separate between three different cases:
Case 1: Assume $|x_1| \geq \delta/\sqrt{2}$**,** $|x_2| \geq \delta/\sqrt{2}$. Then,

$$\begin{aligned}\dot{\mathcal{V}} \leq & -\left(\kappa_1 - \tilde{c}\frac{\sqrt{2}}{\delta} - \tilde{c}\left(1 + (1+\lambda)\left|\theta_2^\star\right|\right)\right)|x_1|^2 \\ & -\left(\kappa_2 - c_2\frac{\sqrt{2}}{\delta} - \tilde{c}\left(1 + \frac{1}{\lambda}\left|\theta_2^\star\right|\right)\right)|x_2|^2\end{aligned}$$

To make the expressions within the parenthesis positive, κ_1 and κ_2 are chosen to satisfy:

$$\kappa_1 > \frac{\tilde{c}\sqrt{2}}{\delta} + \tilde{c}\left(1 + (1+\lambda)\left|\theta_2^\star\right|\right)$$

and

$$\kappa_2 > c_2\frac{\sqrt{2}}{\delta} + \tilde{c}\left(1 + \frac{1}{\lambda}\left|\theta_2^\star\right|\right).$$

where the second inequality is made possible by picking λ sufficiently large.
Case 2: Assume $|x_1| \geq \delta/\sqrt{2}$**,** $|x_2| \leq \delta/\sqrt{2}$. Then,

$$\begin{aligned}\dot{\mathcal{V}} \leq & -\left(\kappa_1 - (c_2 + \tilde{c})\frac{\sqrt{2}}{\delta} - \tilde{c}\left(1 + (1+\lambda)\left|\theta_2^\star\right|\right)\right)|x_1|^2 \\ & -\left(\kappa_2 - \tilde{c}\left(1 + \frac{1}{\lambda}\left|\theta_2^\star\right|\right)\right)|x_2|^2\end{aligned}$$

The following choice of gains, ensures that the expressions within the parenthesis are positive:

$$\kappa_1 > (c_2 + \tilde{c})\frac{\sqrt{2}}{\delta} + \tilde{c}\left(1 + (1+\lambda)\left|\theta_2^\star\right|\right),$$

$$\kappa_2 > \tilde{c}\left(1 + \frac{1}{\lambda}\left|\theta_2^\star\right|\right).$$

Case 3: Assume $|x_1| \leq \delta/\sqrt{2}$, $|x_2| \geq \delta/\sqrt{2}$. Then,

$$\dot{\mathcal{V}} \leq -\left(\kappa_1 - \tilde{c}\left(1 + (1+\lambda)\left|\theta_2^\star\right|\right)\right)|x_1|^2 - \left(\kappa_2 - (c_2 + \tilde{c})\,\frac{\sqrt{2}}{\delta} - \tilde{c}\left(1 + \frac{1}{\lambda}\left|\theta_2^\star\right|\right)\right)|x_2|^2$$

This time it is sufficient to pick κ_1 as:

$$\kappa_1 > \tilde{c}\left(1 + (1+\lambda)\left|\theta_2^\star\right|\right),$$

and κ_2 is chosen to satisfy:

$$\kappa_2 > \frac{(\tilde{c} + c_2)\sqrt{2}}{\delta} + \tilde{c}\left(1 + \frac{1}{\lambda}\left|\theta_2^\star\right|\right).$$

Now, based on the previous calculations, we can pick κ_1 and κ_2 large enough (for instance as the maximum of the previous calculated κ_1 and κ_2 in the three cases), such that for all $|x| \geq \delta$,

$$\dot{\mathcal{V}} \leq -\chi\,|x|^2,$$

for some positive constant χ. From (2.21) and (2.22) we find that

$$\underline{\chi}\,|x|^2 \leq \mathcal{V} \leq \overline{\chi}\,|x|^2,$$

where $\underline{\chi} := \min\{\underline{\kappa}_1, \underline{\kappa}_2\}$ and $\overline{\chi} := \max\{\overline{\kappa}_1, \overline{\kappa}_2\}$. Hence, the inequalities (2.5) and (2.6) of Theorem 2.1, are satisfied with $p = 2$, $\underline{\kappa} = \underline{\chi}$, $\overline{\kappa} = \overline{\chi}$ and $V_\delta = \mathcal{V}$. To show practical stability, i.e. that δ can be diminished at will by conveniently tuning the gains, we notice that $\theta^\star \sim 1/\delta$, such that condition (2.7) holds, and which concludes the proof. ■

A.2 UGPAS

Before we prove Theorem 2.7, we will present the following lemma:

Lemma A.1 *If*

$$L \geq \sum_{i=1}^{N} \alpha_i\left(|x_i|\right) \tag{A.1}$$

where $x_i \in \mathbb{R}$ are subvectors of $x \in \mathbb{R}^n$ and $\alpha_i \in \mathcal{K}$ for all i, then there exists a constant c, such that

$$L \geq \alpha\left(|x|\right) \tag{A.2}$$

where

$$\alpha(s) := c \min_{i \in 1,\ldots,N} \alpha_i(s).$$

Proof. Let

$$S := \left\{ x \quad | \quad L \geq \sum_{i=1}^{N} \alpha_i(|x_i|) \right\}$$

If there are no c such that (A.2) holds, then this would imply the existence of a sequence of vectors in S, $\{r_k\}$, such that

$$\min_{i \in 1,\ldots,N} \alpha_i(|r_k|) \underset{k \to \infty}{\longrightarrow} \infty.$$

(If not, then there would exist a finite constant M, such that

$$\sup \min_{i \in 1,\ldots,N} \alpha_i(|x|) \leq M, \quad x \in S$$

and (A.2) would have been satisfied trivially with $c = L/M$). But if such sequence exists, then none of the functions α_i would be upper bounded, since the minimum over all such functions α_i approached infinity. In addition we have that at least one of the subvectors of the elements of the sequence $\{r_k\}$ approach infinity. But if none of the functions α_i are bounded, and S contains elements where at least one of the subvectors approaches infinity, then

$$\sum_{i=1}^{N} \alpha_i(|x_i|) \longrightarrow \infty$$

and the claim is proved by contradiction. ■

We are now ready to present the proof of Theorem 2.7:

Proof of Theorem 2.7. Due to Young's inequality and Assumption 2.7 we can find the following bound:

$$\begin{aligned} |x_1| \sigma(x_1, x_2, \theta_2^\star) \leq{}& |x_1| + |\theta_2^\star| |x_1|^2 + 2|x_1|^2 + \sum_{q=1}^{Q} |x_2|^{2q} + |x_1|^2 \sum_{q=1}^{Q} |x_2|^{2q} \\ & + |\theta_2^\star| |x_1|^2 \sum_{q=1}^{Q} \tilde{\lambda}_q + |\theta_2^\star| \sum_{q=1}^{Q} \frac{|x_2|^{2q}}{\tilde{\lambda}_q} \\ & + |\theta_2^\star| |x_1|^2 \sum_{q=1}^{Q} \bar{\lambda}_q + |\theta_2^\star| |x_1|^2 \sum_{q=1}^{Q} \frac{|x_2|^{2q}}{\bar{\lambda}_q}, \end{aligned}$$

where $\bar{\lambda}_q$ and $\tilde{\lambda}_q$, $q = 1, .., Q$ are constants. Now, define

$$\mathcal{V} := \ln\left(1 + V_{\delta_1}\right) + \sum_{q=1}^{Q} V_{\delta_2}^{q}.$$

The time derivative of $\mathcal{V}$ is

$$\dot{\mathcal{V}} = \frac{\dot{V}_{\delta_1}}{1 + V_{\delta_1}} + \sum_{q=1}^{Q} q V_{\delta_2}^{(q-1)} \dot{V}_{\delta_2},$$

which along (2.19) and (2.20) have the following bound:

$$\begin{aligned}
\dot{\mathcal{V}} \leq & -\frac{1}{1 + V_{\delta_1}} \Bigg(\kappa_1 \left|x_1\right|^2 - \tilde{c}\left|x_1\right| - 2\tilde{c}\left|x_1\right|^2 \\
& - \tilde{c}\left|\theta_2^{\star}\right| \left|x_1\right|^2 \left(1 + \sum_{q=1}^{Q} \tilde{\lambda}_q + \bar{\lambda}_q\right)\Bigg) \\
& - \sum_{q=1}^{Q} \Bigg(\kappa_2 q \left|x_2\right|^2 V_{\delta_2}^{(q-1)} - c_2 q \left|x_2\right| V_{\delta_2}^{(q-1)} \\
& - \frac{\tilde{c}}{1 + V_{\delta_1}} \left(1 + \left|x_1\right|^2 + \frac{\left|\theta_2^{\star}\right|}{\tilde{\lambda}_q} + \left|x_1\right|^2 \frac{\left|\theta_2^{\star}\right|}{\bar{\lambda}_q}\right) \left|x_2\right|^{2q} \Bigg)
\end{aligned}$$

Due to (2.21), that is $V_{\delta_1} \geq \underline{\kappa}_1 \left|x_1\right|^2$, we find that

$$\frac{\left|x_1\right|^2}{1 + V_{\delta_1}} \leq \frac{1}{\underline{\kappa}_1}$$

and

$$\frac{1}{1 + V_{\delta_1}} \leq 1.$$

By (2.22), we have that

$$\left|x_2\right|^{2q} \leq \frac{\left|x_2\right|^2 V_{\delta_2}^{(q-1)}}{\underline{\kappa}_2^{(q-1)}}$$

we can bound the time derivative of $\mathcal{V}$ in the following way:

$$
\begin{aligned}
\dot{\mathcal{V}} \leq -\frac{1}{1+V_{\delta_1}} \Bigg(& \kappa_1 |x_1|^2 - \tilde{c}|x_1| - 2\tilde{c}|x_1|^2 \\
& - \tilde{c}\,|\theta_2^\star|\,|x_1|^2 \left(1 + \sum_{q=1}^{Q} \tilde{\lambda}_q + \bar{\lambda}_q\right)\Bigg) \\
& - \sum_{q=1}^{Q} \Bigg(\kappa_2 q - \frac{c_2 q}{|x_2|} \\
& - \frac{\tilde{c}}{\underline{\kappa}_2^{(q-1)}} \left(1 + \frac{|\theta_2^\star|}{\tilde{\lambda}_q} + \frac{1}{\underline{\kappa}_1}\left(1 + \frac{|\theta_2^\star|}{\bar{\lambda}_q}\right)\right)\Bigg) |x_2|^2 V_{\delta_2}^{(q-1)}
\end{aligned}
$$

Now, define

$$\delta := \max\{\delta_1, \delta_2\},$$

and notice that

$$|x| \geq \delta \implies \max\{|x_1|, |x_2|\} \geq \frac{\delta}{\sqrt{2}}.$$

We will now separate between three different cases.
Case 1: Assume $|x_1| \geq \delta/\sqrt{2}$**,** $|x_2| \geq \delta/\sqrt{2}$. Then,

$$
\begin{aligned}
\dot{\mathcal{V}} \leq -\frac{1}{1+V_{\delta_1}} \Bigg(& \kappa_1 - \frac{\tilde{c}\sqrt{2}}{\delta} - 2\tilde{c} \\
& - \tilde{c}\,|\theta_2^\star| \left(1 + \sum_{q=1}^{Q} \tilde{\lambda}_q + \bar{\lambda}_q\right)\Bigg) |x_1|^2 \\
& - \sum_{q=1}^{Q} \Bigg(\kappa_2 q - \frac{c_2 q\sqrt{2}}{\delta} \\
& - \frac{\tilde{c}}{\underline{\kappa}_2^{(q-1)}} \left(1 + \frac{|\theta_2^\star|}{\tilde{\lambda}_q} + \frac{1}{\underline{\kappa}_1}\left(1 + \frac{|\theta_2^\star|}{\bar{\lambda}_q}\right)\right)\Bigg) |x_2|^2 V_{\delta_2}^{(q-1)}
\end{aligned}
$$

We choose:

$$\kappa_1 > \frac{\tilde{c}\sqrt{2}}{\delta} + 2\tilde{c} + \tilde{c}\,|\theta_2^\star| \left(1 + \sum_{q=1}^{Q} \tilde{\lambda}_q + \bar{\lambda}_q\right)$$

and

$$\kappa_2 > \frac{1}{q}\left(\frac{c_2 q\sqrt{2}}{\delta} + \frac{\tilde{c}}{\underline{\kappa}_2^{(q-1)}}\left(1 + \frac{|\theta_2^\star|}{\tilde{\lambda}_q} + \frac{1}{\underline{\kappa}_1}\left(1 + \frac{|\theta_2^\star|}{\bar{\lambda}_q}\right)\right)\right),$$

for all $q \in \{1, ..., Q\}$.
Case 2: Assume $|x_1| \geq \delta/\sqrt{2}$**,** $|x_2| \leq \delta/\sqrt{2}$. First notice that

$$\begin{aligned}
\sum_{q=1}^{Q} c_2 q\,|x_2|\,V_{\delta_2}^{(q-1)} &= \frac{|x_1|^2}{1+V_{\delta_1}}\frac{1+V_{\delta_1}}{|x_1|^2}\sum_{q=1}^{Q} c_2 q\,|x_2|\,V_{\delta_2}^{(q-1)} \\
&\leq \frac{|x_1|^2}{1+V_{\delta_1}}\frac{\left(1+\underline{\kappa}_1\,|x_1|^2\right)}{|x_1|^2}\sum_{q=1}^{Q} c_2 q\,|x_2|\left(\overline{\kappa}_2\,|x_2|^2\right)^{(q-1)} \\
&\leq \frac{|x_1|^2}{1+V_{\delta_1}}\left(\frac{2}{\delta^2}+\underline{\kappa}_1\right)\sum_{q=1}^{Q} c_2 q\overline{\kappa}_2^{(q-1)}\,|x_2|^{(2q-1)} \\
&\leq \frac{|x_1|^2}{1+V_{\delta_1}}\left(\frac{2}{\delta^2}+\underline{\kappa}_1\right)\sum_{q=1}^{Q} c_2 q\overline{\kappa}_2^{(q-1)}\frac{\delta^{(2q-1)}}{2^q\sqrt{2}}
\end{aligned}$$

Then,

$$\begin{aligned}
\dot{\mathcal{V}} \leq -\frac{1}{1+V_{\delta_1}}&\left(\kappa_1 - \frac{\tilde{c}\sqrt{2}}{\delta} - \left(\frac{2}{\delta^2}+\underline{\kappa}_1\right)\sum_{q=1}^{Q} c_2 q\overline{\kappa}_2^{(q-1)}\frac{\delta^{(2q-1)}}{2^q\sqrt{2}}\right. \\
&\left. - 2\tilde{c} - \tilde{c}\,|\theta_2^\star|\left(1+\sum_{q=1}^{Q}\tilde{\lambda}_q + \bar{\lambda}_q\right)\right)|x_1|^2 \\
&- \sum_{q=1}^{Q}\left(\kappa_2 q - \frac{\tilde{c}}{\underline{\kappa}_2^{(q-1)}}\left(1+\frac{|\theta_2^\star|}{\tilde{\lambda}_q} + \frac{1}{\underline{\kappa}_1}\left(1+\frac{|\theta_2^\star|}{\bar{\lambda}_q}\right)\right)\right)|x_2|^2\,V_{\delta_2}^{(q-1)}
\end{aligned}$$

The following choice of gains, ensures that the expressions within the parenthesis are positive:

$$\begin{aligned}
\kappa_1 > \frac{\tilde{c}\sqrt{2}}{\delta} &+ \left(\frac{2}{\delta^2}+\underline{\kappa}_1\right)\sum_{q=1}^{Q} c_2 q\overline{\kappa}_2^{(q-1)}\frac{\delta^{(2q-1)}}{2^q\sqrt{2}} \\
&+ 2\tilde{c} + \tilde{c}\,|\theta_2^\star|\left(1+\sum_{q=1}^{Q}\tilde{\lambda}_q + \bar{\lambda}_q\right),
\end{aligned}$$

and

$$\kappa_2 > \frac{\tilde{c}}{q\underline{\kappa}_2^{(q-1)}}\left(1+\frac{|\theta_2^\star|}{\tilde{\lambda}_q}+\frac{1}{\underline{\kappa}_1}\left(1+\frac{|\theta_2^\star|}{\bar{\lambda}_q}\right)\right),$$

for all $q \in \{1, ..., Q\}$.
Case 3: Assume $|x_1| \leq \delta/\sqrt{2}$**,** $|x_2| \geq \delta/\sqrt{2}$. Now, notice that

$$\begin{aligned}\frac{1}{1+V_{\delta_1}}\tilde{c}\,|x_1| &= \frac{1}{1+V_{\delta_1}}\tilde{c}\frac{|x_1|}{|x_2|^2}\,|x_2|^2\\ &\leq \tilde{c}\frac{1}{|x_2|}\,|x_2|^2\\ &\leq \tilde{c}\frac{\sqrt{2}}{\delta}\,|x_2|^2\end{aligned}$$

and therefore

$$\begin{aligned}\dot{\mathcal{V}} \leq &-\frac{1}{1+V_{\delta_1}}\left(\kappa_1 - 2\tilde{c} - \tilde{c}\,|\theta_2|\left(1+\sum_{q=1}^{Q}\tilde{\lambda}_q+\bar{\lambda}_q\right)\right)|x_1|^2\\ &-\left(\kappa_2 - \frac{(\tilde{c}+c_2)\sqrt{2}}{\delta} - \tilde{c}\left(1+\frac{|\theta_2^\star|}{\tilde{\lambda}_1}+\frac{1}{\underline{\kappa}_1}\left(1+\frac{|\theta_2^\star|}{\bar{\lambda}_1}\right)\right)\right)|x_2|^2\\ &-\sum_{q=2}^{Q}\left(\kappa_2 q - \frac{c_2 q\sqrt{2}}{\delta}\right.\\ &\left.-\frac{\tilde{c}}{\underline{\kappa}_2^{(q-1)}}\left(1+\frac{|\theta_2^\star|}{\tilde{\lambda}_q}+\frac{1}{\underline{\kappa}_1}\left(1+\frac{|\theta_2^\star|}{\bar{\lambda}_q}\right)\right)\right)|x_2|^2\,V_{\delta_2}^{(q-1)}\end{aligned}$$

This time it is sufficient to pick κ_1 as:

$$\kappa_1 > 2\tilde{c} + \tilde{c}\,|\theta_2^\star|\left(1+\sum_{q=1}^{Q}\tilde{\lambda}_q+\bar{\lambda}_q\right),$$

and for every $q \in \{1, .., Q\}$, it is sufficient that κ_2 is chosen to satisfy:

$$\kappa_2 > \frac{(\tilde{c}+c_2 q)\sqrt{2}}{\delta} + \frac{\tilde{c}}{\underline{\kappa}_2^{(q-1)}}\left(1+\frac{|\theta_2^\star|}{\tilde{\lambda}_q}+\frac{1}{\underline{\kappa}_1}\left(1+\frac{|\theta_2^\star|}{\bar{\lambda}_q}\right)\right).$$

Now, based on the previous calculations, we can pick κ_1 and κ_2 large enough, such that for all $|x| \geq \delta$,

$$\begin{aligned}\dot{\mathcal{V}} &\leq -\frac{1}{2}\kappa_1 \frac{|x_1|^2}{1+V_{\delta_1}} - \frac{1}{2}\kappa_2 |x_2|^2 \sum_{q=1}^{Q} V_{\delta_2}^{(q-1)} \\ &\leq -\frac{1}{2}\kappa_1 \frac{|x_1|^2}{1+\overline{\kappa}_1 |x_1|^2} - \frac{1}{2}\kappa_2 \sum_{q=1}^{Q} \overline{\kappa}_2^{(q-1)} |x_2|^{2q} \qquad \text{(A.3)}\end{aligned}$$

The right hand side is negative definite and and non-increasing. Define the class $\mathcal{K}$ functions

$$f_1(r) := \frac{1}{2}\kappa_1 \frac{r^2}{1+\bar{\kappa} r^2},$$

and

$$f_2(r) := \frac{1}{2}\kappa_2 \sum_{q=1}^{Q} \bar{\kappa}_2^{(q-1)} r^{2q}.$$

From (A.3), we see that the inequality

$$\dot{\mathcal{V}} \leq -\frac{1}{n_1} \sum_{i=1}^{n_1} f_1(|x_{1i}|) + \sum_{j=1}^{n_2} f_2(|x_{2j}|)$$

also holds, where x_{1i} is element i of the vector $x_1 \in \mathbb{R}^{n_1}$ and x_{2j} is element j of the vector $x_2 \in \mathbb{R}^{n_2}$. As a consequence of Lemma A.1 there is a class $\mathcal{K}$ function α_δ such that for all $x \in \mathbb{R}^n / \bar{\mathcal{B}}_\delta$

$$\dot{\mathcal{V}} \leq -\alpha_\delta(|x|).$$

To apply (Chaillet and Loría, 2008, Lemma 27), the function α_δ is required to be of class $\mathcal{K}_\infty$, but a closer investigation of the proof shows that it is sufficient for α_δ to be non-decreasing. Furthermore, due to the positive definiteness of $\mathcal{V}$, (Khalil, 2002, Lemma 4.3) ensures that for all $x \in \mathbb{R}^n / \bar{\mathcal{B}}_\delta$,

$$\underline{\alpha}_\delta(|x|) \leq \mathcal{V}(t,x) \leq \overline{\alpha}_\delta(|x|)$$

for some class $\mathcal{K}$ function $\underline{\alpha}_\delta, \overline{\alpha}_\delta$. This means that the conditions of (Chaillet and Loría, 2008, Lemma 27) hold, and for any $k > 0$, there exists a C^1 function V and $\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}} \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n / \bar{\mathcal{B}}_\delta$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}\widetilde{\underline{\alpha}}(|x|) &\leq V(t,x) \leq \widetilde{\overline{\alpha}}(|x|) \\ \dot{V}(t,x) &\leq -kV(t,x).\end{aligned}$$

Also, for any $s \in \mathbb{R}_{\geq 0}$, it holds that

$$\underline{\widetilde{\alpha}}^{-1} \circ \widetilde{\overline{\alpha}}\,(s) = \underline{\alpha}_\delta \circ \overline{\alpha}_\delta\,(s)$$

We are now able to apply (Chaillet and Loría, 2008, Lemma 28), so for all $x_0 \in \mathbb{R}^n$ and all $t_0 \in \mathbb{R}_{\geq 0}$ we have that

$$|x\,(t, t_0, x_0)| \leq \underline{\widetilde{\alpha}}_\delta^{-1} \circ \widetilde{\overline{\alpha}}_\delta\,(\delta) + \underline{\widetilde{\alpha}}_\delta^{-1} \left(\widetilde{\overline{\alpha}}_\delta\,(|x_0|)\, e^{-k(t-t_0)}\right)$$

Hence we have that for all $x_0 \in \mathbb{R}^n$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$|x\,(t, t_0, x_0)| \leq \tilde{\delta} + \beta_\delta\,(|x_0|\,, t - t_0)$$

where

$$\begin{aligned} \tilde{\delta} &:= \underline{\widetilde{\alpha}}_\delta^{-1} \circ \widetilde{\overline{\alpha}}_\delta\,(\delta) \\ &= \underline{\alpha}_\delta^{-1} \circ \overline{\alpha}_\delta\,(\delta) \end{aligned}$$

$$\beta_\delta\,(s, t) := \underline{\widetilde{\alpha}}_\delta^{-1} \left(\widetilde{\overline{\alpha}}_\delta\,(s)\, e^{-kt}\right) \in \mathcal{KL}.$$

To show practical stability, i.e. that $\tilde{\delta}$ can be diminished at will by conveniently tuning the gains, we will prove that

$$\begin{aligned} \lim_{\delta \to 0} \underline{\widetilde{\alpha}}^{-1} \circ \widetilde{\overline{\alpha}}\,(s) &= \lim_{\delta \to 0} \underline{\alpha}_\delta^{-1} \circ \overline{\alpha}_\delta\,(\delta) \\ &= 0. \end{aligned}$$

Since $\underline{\alpha}_\delta$ is a class $\mathcal{K}_\infty$ function, independent of δ, it will suffice to prove that

$$\lim_{\delta \to 0} \overline{\alpha}_\delta\,(\delta) = 0 \tag{A.4}$$

For δ small, $\overline{\alpha}_\delta\,(s) \sim \overline{\kappa}_2^q s^{2q}$. Since $\overline{\kappa}_2 \sim 1/\delta$, (A.4) is indeed satisfied, and the conclusion follows. ∎